CONVEX EXTENSION AND LINEAR INEQUALITIES*

BY L. L. DINES

A few years ago, at the Des Moines meeting, it was my privilege to address the Society on the subject *Linear inequalities*. In its simplest form the problem there considered had to do with a system of conditions

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \qquad (i = 1, 2, \cdots, m), \tag{1}$$

the coefficients a_{ij} being given real constants, and the x_j unknowns to be determined so as to satisfy the given conditions. The treatment was entirely analytic, and the aim was to develop a theory dictated by analogy with the theory of linear equations.

Today my purpose is to focus attention on geometric aspects of the theory, and in particular to show its close relationship to a certain geometric notion which in recent years has been useful in many investigations in analysis.

Let us consider an n-dimensional euclidean space $\mathfrak{U}$ of points $u \equiv (u_1, u_2, \cdots, u_n)$. A set of points in $\mathfrak{U}$ is said to be *convex* if the membership of two points $u^{(1)}$ and $u^{(2)}$ in the set implies the membership of all points on the line segment joining $u^{(1)}$ and $u^{(2)}$. A given set may or may not be convex, but any set $\mathfrak{M}$ may be *extended* so as to be convex by adjunction of the necessary points. The resulting set, which may be defined logically as the greatest common subset of all the convex sets which contain $\mathfrak{M}$, will be called the *convex extension* of $\mathfrak{M}$ and denoted by $C(\mathfrak{M})$.

This extended set, under various names and definitions, has been the subject of considerable study, and has been useful in many applications. One may refer to Minkowski's *Geometrie der Zahlen*, 1910; to Carathéodory's paper *Ueber der Variabilitätsbereich der Fourierschen Konstanten* in the Rendiconti del Circolo Matematico di Palermo (vol. 32 (1911)); or to the recent com-

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prehensive work by Bonnesen and Fenchel entitled *Theorie der Konvexen Körper*, 1934, which includes an extensive bibliography.*

Let us note some of the properties of the convex extension $C(\mathfrak{M})$, limiting ourselves for simplicity to the case in which the original set $\mathfrak{M}$ is closed and bounded. The convex extension is easily seen to be closed, bounded, and perfect if $\mathfrak{M}$ contains more than a single point. It has the same dimensionality as the set $\mathfrak{M}$, and if it is truly n-dimensional,† it possesses *inner* points. It possesses two further striking properties, rigorously established by Carathéodory, each of which is indeed characteristic of the convex extension.

The first characterizes the points of $C(\mathfrak{M})$ in terms of the planes ($(n-1)$-flats) in which they lie; and some preliminary definitions and remarks must precede its statement.

A plane ($(n-1)$-flat) in $\mathfrak{U}$ is the locus of points $u \equiv (u_1, u_2, \cdots, u_n)$ satisfying a linear equation

$$\pi(u) \equiv c_0 + \sum_{j=1}^{n} c_j u_j = 0,$$

in which not all the coefficients c are zero. Any such plane determines two *open half-spaces* in $\mathfrak{U}$, consisting respectively of the points u for which $\pi(u) > 0$ and $\pi(u) < 0$. Each of these half-spaces becomes *closed* upon the adjunction of the points of the bounding plane $\pi(u) = 0$. Relative to our given closed and bounded set $\mathfrak{M}$, the plane $\pi(u) = 0$ will be called:

(1) *a bounding plane* (*Schranke*), if the points of $\mathfrak{M}$ are all in the same one of the two *open* half-spaces determined by $\pi(u) = 0$;

* Reference should also be made to the following two very recent articles which of course do not appear in the bibliography of Bonnesen and Fenchel: *Elementare Theorie der konvexen Polyeder*, by Weyl in Commentarii Mathematici Helvetici, vol. 7 (1935); and *Integration of functions with values in a Banach space*, by Garrett Birkhoff in the Transactions of this Society, vol. 38 (1935).

As to terminology, the most commonly used name for what I have called the convex extension is the "konvexe Hülle" (or convex hull) of a set of points. This name, as it is used, seems to me not only inappropriate but indeed misleading. It might very fittingly be used to designate the aggregate of *boundary* points of the convex extension of a closed and bounded set.

† The term truly n-dimensional will be used to describe a set of points which is in an n-dimensional space but in no $(n-1)$-flat of that space.

(2) *a supporting plane* (*Stützebene*), if the points of $\mathfrak{M}$ are all in the same one of the *closed* half-spaces determined by $\pi(u)=0$, and at least one point of $\mathfrak{M}$ is in the plane $\pi(u)=0$;

(3) *a separating plane*, if each open half-space determined by $\pi(u)=0$ contains a point of $\mathfrak{M}$.

We can now state the first of Carathéodory's characteristic properties of $C(\mathfrak{M})$.

PROPERTY C1. The convex extension of $\mathfrak{M}$ consists of those points of $\mathfrak{U}$ through which pass no bounding planes of $\mathfrak{M}$. Furthermore, the *inner* points of the convex extension are those through which pass no supporting planes, and the *boundary* points are those through which pass supporting planes.

This characterization furnishes an ideal geometric representation for the study of linear inequalities. To apply it to our system

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \qquad (i = 1, 2, \cdots, m), \tag{1}$$

we take for the set of points $\mathfrak{M}$, the m points

$$\mathfrak{M}: \qquad (a_{i1}, a_{i2}, \cdots, a_{in}), \qquad (i = 1, 2, \cdots, m),$$

and consider the relation of this point set $\mathfrak{M}$ to the planes

$$\sum_{j=1}^{n} c_j u_j = 0$$

through the origin.

If any one of these planes is a bounding plane of $\mathfrak{M}$, its coefficients $(c_1, c_2, \cdots, c_n)$ (or their negatives) constitute a solution of (1). And conversely, to every solution of (1) there corresponds a bounding plane of $\mathfrak{M}$ through the origin. Hence we have the following theorem.

THEOREM 1. *A necessary and sufficient condition for the existence of a solution of the system of inequalities* (1) *is that the origin* $(0, 0, \cdots, 0)$ *shall not belong to the convex extension of* $\mathfrak{M}$. *The solutions are the sets of coefficients* (*appropriately signed*) *of the bounding planes of* $\mathfrak{M}$ *through the origin.*

The distinction between inner points and boundary points of $C(\mathfrak{M})$ leads to an interpretation of the weaker system of inequalities considered by Minkowski

$$\sum_{j=1}^{n} a_{ij}x_j \geqq 0, \qquad (i = 1, 2, \cdots, m). \tag{2}$$

THEOREM 2. *A necessary and sufficient condition for the existence of a non-trivial* solution of the system of inequalities* (2) *is that the origin shall not be an inner point of the convex extension of* $\mathfrak{M}$. *The solutions are the sets of coefficients* (*appropriately signed*) *of the bounding planes and the supporting planes of* $\mathfrak{M}$ *through the origin.*

The two theorems just stated were obtained in essence by Miss R. W. Stokes in her dissertation published in the Transactions of this Society (1931). The geometric method she used enabled her also to study the character and representation of the solutions of the systems (1) and (2). The so-called fundamental solutions which play an important role in the analytic theory of Minkowski correspond to those supporting planes through the origin which contain $n-1$ points of $\mathfrak{M}$ which lie in no common $(n-2)$-flat. The *general* solution is a linear combination of the fundamental solutions, with positive coefficients in the case of (1) and non-negative coefficients in the case of (2).†

Recalling the manner in which Theorems 1 and 2 followed from a property of the convex extension $C(\mathfrak{M})$, we note that it is not essential that the number of points in $\mathfrak{M}$ be finite, that is, that the number of inequalities be finite as in (1) and (2). We may with equal ease consider systems

$$\sum_{j=1}^{n} f_j(p)x_j > 0, \qquad (p \text{ on the range } \mathfrak{P}), \tag{1p}$$

and

* Since the system (2) always admits the solution $x_j=0, (j=1, 2, \cdots, n)$, this will be called the *trivial* solution. The system (2) may admit other solutions for which the *equality* sign holds for every i. This will be the case if and only if the set $\mathfrak{M}$ lies in an $(n-1)$-flat through the origin. This is consistent with the theorem, since in that case the set $C(\mathfrak{M})$ has no inner points.

† The geometric method of approach has also been used by Haar, *Ueber lineare Ungleichungen*, Szeged Acta, sectio scientiarum mathematicarum, vol. 2 (1924); Fujiwara, *On the system of linear inequalities and linear integral inequality*, Proceedings of the Imperial Academy, vol. 4 (1928); Dines and McCoy, *On linear inequalities*, Transactions of the Royal Society of Canada, vol. 27 (1933).

$$\sum_{j=1}^{n} f_j(p)x_j \geqq 0, \qquad (p \text{ on the range } \mathfrak{P}), \tag{2p}$$

where $\mathfrak{P}$ is any class of elements p, the functions $f_j(p)$ are real-valued bounded functions on the range $\mathfrak{P}$, provided the set of points $\mathfrak{M}$ represented parametrically in the space $\mathfrak{U}$ by

$$\mathfrak{M}: \qquad u_1 = f_1(p),\ u_2 = f_2(p),\ \cdots,\ u_n = f_n(p), \qquad (p \text{ on } \mathfrak{P}),$$

is a closed set. The natural generalizations of Theorems 1 and 2 follow immediately.* However the following equivalent statement is more interesting from the point of view of analysis.

THEOREM 3. *A necessary and sufficient condition that every linear combination*

$$\sum_{j=1}^{n} c_j f_j(p), \qquad (p \text{ on range } \mathfrak{P}) \tag{3}$$

of the functions $f_j(p)$ shall change sign or vanish on $\mathfrak{P}$ is that the origin $(0, 0, \cdots, 0)$ shall belong to the convex extension of $\mathfrak{M}$. A necessary and sufficient condition that every linear combination (3) *shall change sign on $\mathfrak{P}$ is that the origin shall be an inner point of the convex extension of $\mathfrak{M}$.*

So far our discussion has been based on the first of Carathéodory's characteristic properties of the convex extension. Let us now consider his second property.

PROPERTY C2. The convex extension of $\mathfrak{M}$ consists of those points of $\mathfrak{U}$ which can be the centroids of positive mass distributions (of total mass unity) at suitably chosen points of $\mathfrak{M}$.

Carathéodory showed that only a finite number (at most $n+1$) of points of $\mathfrak{M}$ are necessary thus to determine any point of $C(\mathfrak{M})$ as a centroid. Hence Property C2 can be expressed in analytic terms by saying that $C(\mathfrak{M})$ consists of those points of $\mathfrak{U}$ whose coordinates admit a representation of form

$$u_j = \sum_{i=1}^{r} \mu_i u_{ij}, \qquad (j = 1, 2, \cdots, n), \tag{4}$$

where

$$(u_{i1}, u_{i2}, \cdots, u_{in}), \qquad (i = 1, 2, \cdots, r), \tag{5}$$

* Dines and McCoy, loc. cit., p. 59.

is a set of points of $\mathfrak{M}$, and

$$\mu_i > 0, \qquad \sum_{i=1}^{r} \mu_i = 1.$$

The representation (4) of a given point of $C(\mathfrak{M})$ is of course not in general unique, and the possibility of variation is of interest, particularly in seeking a distinction between inner points and boundary points. The inner points are characterized by the fact that they admit such representation in terms of a *truly n-dimensional* subset of $\mathfrak{M}$. Remembering that the simplest set of this sort consists of $n+1$ points, one is tempted to associate that number with the representation of inner points. But such an association is erroneous. It is true that a point u which admits the representation (4) in terms of a truly n-dimensional subset of $n+1$ points of $\mathfrak{M}$ is an inner point of $C(\mathfrak{M})$, indeed of the n-dimensional simplex having these points as vertices. But not every inner point of $C(\mathfrak{M})$ necessarily admits such representation. For example, if $n=2$ and $\mathfrak{M}$ consists of the vertices of a square, the center of the square is an inner point of $C(\mathfrak{M})$ but it cannot be given the suggested representation in terms of *three* vertices. The following general statement can be made. An inner point of $C(\mathfrak{M})$ can be represented in terms of a truly n-dimensional subset which consists of not more than $2n$ points of $\mathfrak{M}$.*

For finite sets $\mathfrak{M}$, the inner points of $C(\mathfrak{M})$ are characterized in a slightly different manner in the following theorem, of which we shall see an application and an interesting analog for a certain type of infinite sets.

THEOREM 4. *If $\mathfrak{M}$ is a truly n-dimensional finite set of points, the inner points of the convex extension $C(\mathfrak{M})$ are precisely those for which all points of $\mathfrak{M}$ may be included in the set* (5) *of the representation* (4).

The proof, which is not so simple as one might expect, is quite similar to that of Theorem 12 in the paper by Dines and McCoy.

For infinite sets $\mathfrak{M}$, one is naturally led to attempt the representation of the points of $C(\mathfrak{M})$ by some of the infinite summation processes. Let us suppose, for instance, that $\mathfrak{M}$ consists of the points of a continuous arc defined parametrically by

* Dines and McCoy, loc. cit., pp. 61–63.

$$u_1 = f_1(x),\ u_2 = f_2(x),\ \cdots,\ u_n = f_n(x), \qquad (a \leqq x \leqq b). \tag{6}$$

The obvious suggestion from analogy is that the points u of $C(\mathfrak{M})$ should be expressible in the form

$$u_j = \int_a^b \mu(x) f_j(x)\,dx, \qquad (j = 1, 2, \cdots, n), \tag{7}$$

the function $\mu(x)$ being non-negative and such that

$$\int_a^b \mu(x)\,dx = 1.$$

But caution is necessary here. It is easily seen that every point $\bar{u}$ expressible in form (7) belongs to $C(\mathfrak{M})$. For otherwise there would be a bounding plane

$$\sum_{j=1}^{n} c_j(u_j - \bar{u}_j) = 0$$

through $\bar{u}$, such that

$$\sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j) > 0, \qquad (a \leqq x \leqq b);$$

whence, on multiplying both sides by $\mu(x)$ and integrating, one would have the contradiction

$$\sum_{j=1}^{n} c_j(\bar{u}_j - \bar{u}_j) > 0.$$

But the converse is not true, as the following simple example will show. Let the arc in question be the circular quadrant

$$u_1 = \cos\frac{\pi}{2}x, \qquad u_2 = \sin\frac{\pi}{2}x, \qquad (0 \leqq x \leqq 1).$$

The point $(u_1, u_2) = (1, 0)$ belongs to $C(\mathfrak{M})$. But there is no non-negative function $\mu(x)$ such that

$$1 = \int_0^1 \mu(x)\cos\frac{\pi}{2}x\,dx, \qquad 0 = \int_0^1 \mu(x)\sin\frac{\pi}{2}x\,dx,$$

$$\int_0^1 \mu(x)\,dx = 1.$$

Obviously the difficulty here is due to the fact that (1, 0) is a boundary point of $C(\mathfrak{M})$. If we consider only *inner* points, the following interesting analog of Theorem 4 is valid.

THEOREM 5.* *If the continuous arc* (6) *does not lie in an* $(n-1)$*-flat, the inner points of its convex extension are precisely those points* u *whose coordinates can be expressed in the form*

$$u_j = \int_a^b \mu(x) f_j(x)\,dx, \qquad (j = 1, 2, \cdots, n), \tag{8}$$

where $\mu(x)$ *is continuous, and*

$$\int_a^b \mu(x)\,dx = 1, \qquad \mu(x) > 0, \qquad (a \leqq x \leqq b).$$

But in view of the nature of the problem, one can hardly overlook the fact that the Stieltjes integral is the ideal means for representing points of the convex extension. Relative to the continuous arc (6) we have the following theorem, the first part of which is due to F. Riesz.†

THEOREM 6. *The convex extension of the continuous arc represented parametrically by* (6) *consists of those points* u *which admit a representation*

$$u_j = \int_a^b f_j(x)\,d\alpha(x), \qquad (j = 1, 2, \cdots, n), \tag{9}$$

where $\alpha(x)$ *is a monotonic non-decreasing function such that*

$$\int_a^b d\alpha(x) = 1. \tag{10}$$

If the arc does not lie in an $(n-1)$*-flat, the inner points of the convex extension are those which admit the representation* (9) *with* $\alpha(x)$ *a monotonic increasing function satisfying* (10).

* Proved by Schoenberg, this Bulletin, vol. 39 (1933). It appears, however, that Schoenberg's hypothesis that the functions $f_j(x)$ be linearly independent should be replaced by the more restrictive condition that no linear combination of these functions be constant, which is equivalent to our hypothesis.

Another proof may be found in Fenchel's *Geschlossene Raumkurven mit vorgeschriebenem Tangentenbild*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 40 (1930).

† Annales de l'École Normale, vol. 28, pp. 56–57.

The proof, in view of our previous discussion, is quite simple. From well known properties of the Stieltjes integral it follows that the finite sum in formula (4), which represents any point of the convex extension, can be replaced by a Stieltjes integral of form (9) with $\alpha(x)$ satisfying (10). It suffices to take for $\alpha(x)$ a step function with jumps of magnitude μ_i at suitably chosen points.

Conversely, if a point $\bar{u}$ admits the representation (9) with $\alpha(x)$ monotonic non-decreasing and satisfying (10), the point belongs to the convex extension. For otherwise there would pass through it a bounding plane

$$\sum_{j=1}^{n} c_j(u_j - \bar{u}_j) = 0$$

such that

$$\sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j) > 0, \qquad (a \leqq x \leqq b). \tag{11}$$

If now we denote the left side of (11) by $F(x)$, so that

$$F(x) \equiv \sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j), \qquad (a \leqq x \leqq b),$$

we obtain a contradiction in the Stieltjes integral

$$\int_a^b F(x)d\alpha(x).$$

Since $F(x)$ is positive, this integral must be positive, while its evaluation by use of (9) (with $u = \bar{u}$) and (10) shows it to be zero.

To prove the second part of the theorem, we observe that formula (8), which will represent any *inner* point of the convex extension, can be thrown into the form (9) by taking

$$\alpha(x) = \int_a^x \mu(x)dx, \qquad (a \leqq x \leqq b).$$

Since $\mu(x)$ is positive and continuous, $\alpha(x)$ will be monotonic increasing, and it will satisfy (10) in view of the analogous condition on $\mu(x)$.

Conversely, if a point $\bar{u}$ admits a representation (9) with $\alpha(x)$ monotonic increasing and satisfying (10), then it must be an *inner* point of the convex extension. For if it were a boundary point there would pass through it a supporting plane

$$\sum_{j=1}^{n} c_j(u_j - \bar{u}_j) = 0$$

such that

$$F(x) \equiv \sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j) \geqq 0, \quad (a \leqq x \leqq b),$$

the *inequality* certainly holding for at least one value of x since the arc cannot lie in the supporting $(n-1)$-flat. If now we consider the Stieltjes integral of $F(x)d\alpha(x)$, a simple argument based on the properties of $F(x)$ and $\alpha(x)$ leads to the contradictory conclusions that this integral must be both positive and zero. The contradiction completes the proof of the theorem.

We interpreted the first Carathéodory property of the convex extension in terms of linear inequalities. The second property admits an even more obvious interpretation in terms of linear equations. From our discussion the following conclusions result.*

THEOREM 7. *A necessary and sufficient condition that the system of n linear homogeneous equations in m unknowns*

$$\sum_{i=1}^{m} a_{ij}\mu_i = 0, \qquad (j = 1, 2, \cdots, n), \tag{12}$$

admit a non-negative (and non-trivial) solution $(\mu_1, \mu_2, \cdots, \mu_m)$ *is that, in an n-dimensional euclidean space containing the set of points*

$$\mathfrak{M}: \qquad (a_{i1}, a_{i2}, \cdots, a_{in}), \qquad (i = 1, 2, \cdots, m),$$

the origin shall belong to the convex extension of that set. If the set of points $\mathfrak{M}$ *is truly n-dimensional, a necessary and sufficient condition that the system* (12) *admit a positive solution* $(\mu_1, \mu_2, \cdots, \mu_m)$ *is that the origin be an inner point of the convex extension of* $\mathfrak{M}$.

* For simplicity we limit our statements to the special case of homogeneous equations. The alterations to be made in the non-homogeneous case will be obvious.

THEOREM 8. *A necessary and sufficient condition that the system of Stieltjes integral equations*

$$\int_a^b f_j(x)d\alpha(x) = 0, \qquad (j = 1, 2, \cdots, n), \tag{13}$$

in which the functions $f_j(x)$ are continuous, admit a monotonic non-decreasing (and non-trivial) solution $\alpha(x)$ is that the origin belong to the convex extension of the arc

$$u_1 = f_1(x), u_2 = f_2(x), \cdots, u_n = f_n(x), \qquad (a \leqq x \leqq b).$$

If this arc does not lie in an $(n-1)$-flat, a necessary and sufficient condition that the system (13) *admit a monotonic increasing solution $\alpha(x)$ is that the origin be an inner point of the convex extension.*

And this last condition is also necessary and sufficient for the existence of a positive continuous solution $\mu(x)$ of the system of integral equations

$$\int_a^b f_j(x)\mu(x)dx = 0, \quad (j = 1, 2, \cdots, n).$$

In our discussion we have noted analytic interpretations of each of Carathéodory's two characteristic properties of the convex extension. The equivalence of these two properties of course implies the logical equivalence of the analytic interpretations. Each instance of such equivalence yields a purely analytic theorem. Let us note a few.

THEOREM 9. *A necessary and sufficient condition that the system of linear inequalities*

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \qquad (i = 1, 2, \cdots, m),$$

admit a solution $(x_1, x_2, \cdots, x_n)$ is that the adjoint system of linear equations

$$\sum_{i=1}^{m} a_{ij}y_i = 0, \qquad (j = 1, 2, \cdots, n), \tag{14}$$

admit no non-negative solution $(y_1, y_2, \cdots, y_m)$ other than the trivial zero solution.

A necessary and sufficient condition that the system

$$\sum_{j=1}^{n} a_{ij}x_j \geqq 0, \qquad (i = 1, 2, \cdots, m),$$

admit a solution $(x_1, x_2, \cdots, x_n)$ *which does not annul all the left members, is that the adjoint system of linear equations* (14) *admit no positive solution* $(y_1, y_2, \cdots, y_m)$.

These two results were first stated by me in approximately this form in a note in the Annals of Mathematics in 1926. However, W. B. Carver had essentially obtained the first in a paper in the Annals of 1922. And, unfortunately unknown to either of us, Stiemke had obtained the essence of both in the Mathematische Annalen of 1915. All of the purely analytic proofs were quite complicated.

From our results concerning the convex extension of a continuous arc we draw the following conclusions.

THEOREM 10. *If the n functions* $f_j(x)$ *are continuous and linearly independent on the interval* $(a \leqq x \leqq b)$, *a necessary and sufficient condition that the system of Stieltjes integral equations*

$$\int_a^b f_j(x)d\alpha(x) = 0, \qquad (j = 1, 2, \cdots, n), \tag{15}$$

admit a monotonic non-decreasing solution $\alpha(x)$ (*other than a constant*) *is that every linear combination*

$$\sum_{j=1}^{n} c_j f_j(x) \tag{16}$$

shall vanish somewhere on the interval $(a \leqq x \leqq b)$.

A necessary and sufficient condition that the system (15) *admit a monotonic increasing solution* $\alpha(x)$ *is that every linear combination* (16) *shall change sign on* $(a \leqq x \leqq b)$.

The last condition is likewise necessary and sufficient for the existence of a positive continuous solution $\mu(x)$ *of the system*

$$\int_a^b f_j(x)\mu(x)dx = 0, \qquad (j = 1, 2, \cdots, n).$$

The third part of this theorem I stated and proved in the Transactions of 1928, by what now appears to be a very cumbersome method. It was obtained by the present simple argument

by Schoenberg in the Bulletin of 1933. This type of argument seems first to have been applied to problems of this sort by Kakeya and Fujiwara in various papers in the Japanese journals between 1914 and 1930, though unfortunately some inaccuracies mar their results.

The theorems which we have just obtained may perhaps be described in a general way as *matrix-free* theorems concerning adjoint systems of linear conditions. Two adjoint systems arise from the same matrix. The properties of the matrix determine the nature of the solutions of each system. But once the characterization has been established, the matrix may be eliminated from consideration, and there results a relationship between the natures of the solutions of the adjoint systems.

In our study, the matrix has been interpreted as a set of points and we have been interested in the relationship of its convex extension to the origin of coordinates. Through the two Carathéodory properties this relationship is translated into equivalent properties of the two adjoint linear systems.

Of course the methods and results we have described invite extension and generalization in various directions. We have seen that no particular difficulty is presented by the *number* of conditions in one of the adjoint systems, the one associated with the number of points in the set $\mathfrak{M}$. However, such is not the case with the other system, in which the number of conditions coincides with the dimensionality n of the euclidean space $\mathfrak{U}$. Special instances in which the number of conditions in this system is infinite have been considered by F. Riesz, Schoenberg, McCoy, and myself. But a most tempting generalization would be a theory of the convex extension of a set of points in a space more general than the ordinary euclidean n-space. Contributions in this direction have been made by H. Kneser and Steinitz for a projective space, Keller for Hilbert space, Ascoli and Mazur for general linear spaces, Menger for general metric spaces, and Whitehead for the geometry of paths. But time does not permit a consideration of any of these on this occasion. I must close by thanking you most sincerely for the privilege of addressing you, and for your courteous attention.

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A REMARK ON THE ODD SCHLICHT FUNCTIONS

BY M. S. ROBERTSON

Let (S) denote the class of analytic functions

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1}$$

regular and univalent or schlicht for $|z| < 1$, and (U) the subclass of odd schlicht functions

$$\phi(z) = [f(z^2)]^{1/2} = z + b_3 z^3 + b_5 z^5 + \cdots . \tag{2}$$

If $\phi(z)$ is real on the real axis, it has been shown* by J. Dieudonné that for all n

$$|b_{2n-1}| + |b_{2n+1}| \leqq 2, \qquad |b_3| \leqq 1. \tag{3}$$

This is not known to be true in the case where the coefficients are complex except for $n=1$. For complex coefficients it is known† that

$$|b_3| \leqq 1, \qquad |b_5| \leqq e^{-2/3} + \frac{1}{2} \qquad (> 1), \tag{4}$$

from which we could conclude only that

$$|b_3| + |b_5| \leqq \frac{3}{2} + e^{-2/3} \qquad (> 2).$$

It is the purpose of this paper to establish the inequality (3) for $n=2$ for the case when the coefficients are complex numbers; and to show further that

$$\frac{|b_3| + |b_5|}{2} \leqq \left(\frac{|b_3|^2 + |b_5|^2}{2} \right)^{1/2} \leqq 1, \tag{5}$$

$$|a_3| \leqq 1 + |b_3|^2 + |b_5|^2 \leqq 3. \tag{6}$$

* See J. Dieudonné, Annales de l'Ecole Normale Supérieure, vol. 48 (1931), p. 318.

† See M. Fekete and G. Szegö, Journal of the London Mathematical Society, vol. 8 (1933), pp. 85–89.

The inequality $|a_3| \leqq 3$ is well known,* but the second half of the inequality (6) is new, as far as the author is aware.

Since

$$a_n = \sum_{k=1}^{n} b_{2k-1} b_{2(n-k)+1}, \qquad b_1 = 1,$$

we have by Schwarz's inequality

$$|a_n| \leqq \sum_{k=1}^{n} |b_{2k-1}|^2, \tag{7}$$

and in particular,

$$|a_3| \leqq 1 + |b_3|^2 + |b_5|^2.$$

It is known† that there exists an absolute constant A such that $|b_{2n+1}| \leqq A$ for all n. The conjecture of Paley and Littlewood that $A = 1$ was found to be false by the example of Fekete and Szegö, who demonstrated the existence of an odd function univalent for $|z| < 1$ for which $|b_5| = e^{-2/3} + 1/2 > 1$. We wish to point out that a weaker statement of the conjecture that $|b_{2n+1}| \leqq 1$ might conceivably be true, namely, that

$$\sum_{k=1}^{n} |b_{2k-1}|^2 \leqq n, \qquad b_1 = 1. \tag{8}$$

If this weaker conjecture is correct, then by (7) the well known conjecture of L. Bieberbach, $|a_n| \leqq n$, would also be true. To substantiate the weaker conjecture (8), we shall demonstrate here that (8) is true for $n = 3$. It is already known to be true for $n = 1, 2$ by (4).

The method used is that of Fekete and Szegö,‡ who employ the representation of the coefficients of (1) obtained by K. Löwner.§ Denoting a continuous function of absolute value unity by $k(t)$, we have the following representation which Löwner obtained for the coefficients:

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* See K. Löwner, Mathematische Annalen, vol. 89 (1923), pp. 103–121.

† See R. Paley and J. Littlewood, Journal of the London Mathematical Society, vol. 7 (1932), pp. 167–169.

‡ See M. Fekete and G. Szegö, loc. cit.

§ See K. Löwner, loc. cit.

$$a_2 = -2\int_0^\infty k(t)e^{-t}dt,$$

$$a_3 = 4\left[\int_0^\infty k(t)e^{-t}dt\right]^2 - 2\int_0^\infty k^2(t)e^{-2t}dt,$$

$$b_3 = \frac{a_2}{2} = -\int_0^\infty k(t)e^{-t}dt,$$

$$b_5 = \frac{a_3}{2} - \frac{a_2^2}{8} = \frac{3}{2}\left[\int_0^\infty k(t)e^{-t}dt\right]^2 - \int_0^\infty k^2(t)e^{-2t}dt.$$

Let

$$b_5 = |b_5|e^{2i\beta} \ (\beta \text{ real}), \quad b_3 = |b_3|e^{i(\alpha+\beta)} \ (\alpha \text{ real}), \quad k(t)e^{-i\beta} = e^{i\theta(t)},$$

where $\theta(t)$ is real and continuous save for a finite number of points. Then

$$|b_5| = \Re\left\{\left[\frac{3}{2}\int_0^\infty e^{-t}\cdot e^{i\theta(t)}dt\right]^2 - \int_0^\infty e^{-2t}\cdot e^{2i\theta(t)}dt\right\}$$

$$= \frac{3}{2}\left[\int_0^\infty e^{-t}\cos\theta(t)dt\right]^2 - \frac{3}{2}\left[\int_0^\infty e^{-t}\sin\theta(t)dt\right]^2$$

$$- 2\int_0^\infty e^{-2t}\cos^2\theta(t)dt + \frac{1}{2},$$

$$|b_3|^2 = \left[\int_0^\infty e^{-t}\cos\{\theta(t)-\alpha\}dt\right]^2$$

$$+ \left[\int_0^\infty e^{-t}\sin\{\theta(t)-\alpha\}dt\right]^2.$$

Since the left-hand side of this equation is independent of α, we have

$$|b_3|^2 = \left[\int_0^\infty e^{-t}\cos\theta(t)dt\right]^2 + \left[\int_0^\infty e^{-t}\sin\theta(t)dt\right]^2.$$

Let x denote the non-negative real root of the equation

$$\left(x+\frac{1}{2}\right)e^{-2x} = \int_0^\infty e^{-2t}\cos^2\theta(t)dt.$$

Then, by the theorem of Valiron-Landau,* we have

$$\left| \int_0^\infty e^{-t} \cos \theta(t) dt \right| \leqq (x+1)e^{-x}.$$

Let

$$A = A(x) \equiv \left| \int_0^\infty e^{-t} \sin \theta(t) dt \right| \leqq 1.$$

It follows that $|b_3|^2 + |b_5|^2 \leqq P(x)$, where

$$\begin{aligned} P(x) &\equiv \left[\frac{3}{2}(x+1)^2 e^{-2x} - \frac{3}{2}A^2 - 2\left(x+\frac{1}{2}\right)e^{-2x} + \frac{1}{2} \right]^2 \\ &\quad + \left[(x+1)^2 e^{-2x} + A^2 \right] \\ &= \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3A^2}{2} \right]^2 \\ &\quad + (x+1)^2 e^{-2x} + A^2. \end{aligned}$$

CASE 1. Suppose

$$0 \leqq A^2 \leqq \frac{2}{3}\left[(3x^2+2x+1)e^{-2x} + \frac{1}{3} \right].$$

Then

$$P(x) \leqq \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} \right]^2 + (x+2)^2 \cdot e^{-2x}.$$

The maximum of the right-hand side of this inequality is 2 and occurs for $x=0$. Hence in this case

$$|b_3|^2 + |b_5|^2 \leqq P(x) \leqq 2.$$

CASE 2. Suppose

$$\begin{aligned} \frac{2}{3}\left[(3x^2+2x+1)e^{-2x} + \frac{1}{3} \right] &\leqq A^2 \\ &\leqq \frac{1}{3}\left[(3x^2+2x+1)e^{-2x} + 1 \right]. \end{aligned}$$

* See E. Landau, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634, especially pp. 630–632.

Since $|b_3|^2 \leqq 1$, we have, within the range of A^2 in this case,

$$|b_3|^2 + |b_5|^2 \leqq P(x) \leqq 1 + \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3A^2}{2}\right]^2$$
$$\leqq 1 + \left[\frac{1-3(3x^2+2x+1)e^{-2x}}{6}\right]^2$$
$$< 2, \qquad \text{(for all } x \geqq 0\text{)}.$$

CASE 3. Suppose

$$\frac{1}{3}\left[(3x^2+2x+1)e^{-2x}+1\right] \leqq A^2 \leqq 1.$$

Then

$$|b_3|^2 + |b_5|^2 \leqq P(x) \leqq 1 + \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3A^2}{2}\right]^2$$
$$\leqq 1 + \left[(3x^2+2x+1)\frac{e^{-2x}}{2} + \frac{1}{2} - \frac{3}{2}\right]^2$$
$$\leqq 2, \qquad \text{(for all } x \geqq 0\text{)}.$$

Since these cases exhaust all those possible, we have

$$|b_3|^2 + |b_5|^2 \leqq 2,$$

and the equality sign occurs for the function $z/(1-e^{i\alpha}z^2)$.

An application of Schwarz's inequality gives also

$$|b_3| + |b_5| \leqq 2\left(\frac{|b_3|^2+|b_5|^2}{2}\right)^{1/2} \leqq 2.$$

THE UNIVERSITY OF CHICAGO AND YALE UNIVERSITY

ON THE ANALOG FOR DIFFERENTIAL EQUATIONS OF THE HILBERT-NETTO THEOREM

BY H. W. RAUDENBUSH, JR.

The Hilbert-Netto theorem for polynomials* has an analog, obtained by J. F. Ritt,† for differential polynomials.‡ By a differential polynomial is meant a polynomial in a finite number of unknown functions $y_1, \cdots, y_n$ of the independent variable x and a certain number of their derivatives. To any finite set of differential polynomials $F_1, \cdots, F_r$ there corresponds a finite system of ordinary algebraic differential equations $F_1=0, \cdots, F_r=0$. It is supposed that the coefficients of the F_i are all elements of a field $\mathfrak{F}$ of differentiable functions of x which is closed with respect to differentiation. Let G be any differential polynomial with coefficients in $\mathfrak{F}$ such that $G=0$ has every solution§ of the system. If we denote by Σ the totality of differential polynomials that are linear combinations of the F_i and a certain number of their derivatives with differential polynomials with coefficients in $\mathfrak{F}$ as coefficients, then Ritt's theorem is that some power of G is in Σ.

We show in the following that this is as far in this direction as the analogy between the theories of polynomials and differential polynomials extends. For our result is that, contrary to the easy conjecture, it is not generally true that for a given system there exists a single positive integer ρ such that any differential polynomial G, defined as above for that system, has the power G^ρ in the corresponding Σ. We give an example of a system for which there exist G's with arbitrarily high powers not in Σ. This result shows, in particular, why the ideal theory of

* See B. L. van der Waerden, *Moderne Algebra*, vol. 2, p. 66.

† J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, Colloquium Publications of this Society, vol. 14. (Cited as α.)

‡ J. F. Ritt, *Systems of algebraic differential equations*, Annals of Mathematics, (2), vol. 36 (1935), p. 293. The term "differential polynomial" is equivalent to the terms "differential form" or "form" used in α.

§ See α for a precise definition. For an abstract formulation, see H. W. Raudenbush, Jr., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), pp. 361–368.

differential polynomials as far as it has been developed is marked by the absence of primary ideals.*

In our example, the finite system consists of the single equation $F=y^3=0$. The differential polynomials G, such that $G=0$ has every solution of $F=0$, are simply those differential polynomials in y that have no term free of y and its derivatives. We suppose that there exists a single positive integer ρ, such that for any G, the power G^ρ is in Σ, and show that this assumption leads to a contradiction.

The weight of a power product in y and a certain number of its derivatives is defined as the sum of the products of the order of each derivative present and the degree to which it appears. The totality of differential polynomials whose terms are all of degree ρ and weight w, where w is to be fixed later, we shall denote by Σ_w^ρ.

A linear basis with respect to $\mathfrak{F}$ for Σ_w^ρ may be obtained by taking all products consisting of F or some one of its derivatives d^iF/dx^i, $i \leqq w$, multiplied by a power product in y and its derivatives of degree $\rho-3$ and weight w or $w-i$, respectively. The number of differential polynomials in a set of linearly independent differential polynomials of Σ_w^ρ is consequently not greater than the number of distinct power products of y and its derivatives of degree $\rho-3$ and weight w or less. An upper bound to the number is $(w+1)^{\rho-3}+w^{\rho-3}+(w-1)^{\rho-3}+\cdots+1$, or consequently and more simply, $(w+1)^{\rho-2}$.

We consider now the power products in y and its derivatives of degree ρ and weight w. Suppose, for simplicity, that w is divisible by ρ. Then their number is greater than $w^{\rho-1}/(\rho^{\rho-1}(\rho-1)!)$. This lower bound is obtained by noticing that if we fill the first places in the product with y or any of its derivatives of order less than w/ρ, then it is always possible to fill the last place with a derivative of order greater than w/ρ to make a power product of degree ρ and weight w. The number of permutations with repetitions of the w/ρ things (y and its derivatives of order less than w/ρ) taken $\rho-1$ at a time is $(w/\rho)^{\rho-1}$. Not more than $(\rho-1)!$ of these permutations can give a single combination, that is, a single power product. Now since these power products form a linearly independent set with re-

* See Raudenbush, loc. cit.

spect to $\mathfrak{F}$ and since for a certain large w, their number exceeds $(w+1)^{\rho-2}$, we conclude that, for a certain large w, Σ_w^ρ does not contain all power products of degree ρ and weight w.

Under our assumption, on the other hand, Σ would contain all differential polynomials of the form $(a_0y+a_1dy/dx+\cdots+a_wd^wy/dx^w)^\rho$, where the a_i are arbitrary elements of $\mathfrak{F}$. It is easily seen that each power product of degree ρ and weight w is a linear combination of certain of these differential polynomials and hence is in Σ_w^ρ. This contradiction shows that for the system $F=y^2=0$ no integer ρ exists or, in other words, differential polynomials G exist having arbitrarily large powers not in Σ.

YALE UNIVERSITY

NOTE ON THE GALERKIN AND PAPKOVITCH STRESS FUNCTIONS

BY R. D. MINDLIN

1. *Introduction.* H. M. Westergaard* has given a useful interpretation of the Galerkin stress functions† as the components of a vector function satisfying a fourth order equation. From the Galerkin vector, P. F. Papkovitch‡ has developed a new solution of the three-dimensional elasticity equations for a homogeneous, isotropic solid in terms of harmonic functions. The same solution has been given by H. Neuber.§

Some interesting aspects of the Galerkin and Papkovitch functions may be observed when they are approached from a consideration of Helmholtz's theorem. In so doing, it is found that these functions may be reached by a direct analytical process and that they are connected through simple functional rela-

* H. M. Westergaard, this Bulletin, vol. 41 (1935), p. 695.

† B. Galerkin, Comptes Rendus, vol. 190 (1930), p. 1047. See also Todhunter and Pearson, *History of Elasticity*, vol. 2, part 2, pp. 268–270.

‡ P. F. Papkovitch, Comptes Rendus, vol. 195 (1932), pp. 513, 754. J. N. Goodier calls attention to Todhunter and Pearson, loc. cit., vol. 2, part 2, p. 373.

§ H. Neuber, Zeitschrift für angewandte Mathematik und Mechanik, vol. 14 (1934), p. 203.

tions with each other and with the potentials of the dilatation and rotation.

2. *The Helmholtz Transformation.* It is known that the displacement which gives rise to the dilatation is the gradient of a scalar potential function and the displacement which produces the rotation is the curl of a solenoidal vector potential function, so that the displacement may be expressed by

$$\rho = \text{grad } \phi + \text{curl } \boldsymbol{S}, \tag{1}$$

where ρ is the vector displacement, ϕ is a scalar potential and $\boldsymbol{S}$ is a vector potential such that div $\boldsymbol{S}=0$. This transformation is commonly associated with the name of Helmholtz.

3. *Equilibrium Equation.* In vector form, the equilibrium equation in terms of the displacement is

$$\Delta\rho + \frac{1}{1-2\mu}\,\text{grad div } \rho = -\frac{\boldsymbol{K}}{G}, \tag{2}$$

where Δ is Laplace's operator, μ is Poisson's ratio, $\boldsymbol{K}$ is the vector body force per unit of volume, and G is the modulus of rigidity.

Combining equations (1) and (2) and writing

$$\alpha = \frac{2(1-\mu)}{1-2\mu},$$

we obtain the equilibrium equation in terms of the potential functions ϕ and $\boldsymbol{S}$:

$$\Delta[\alpha\,\text{grad } \phi + \text{curl } \boldsymbol{S}] = -\frac{\boldsymbol{K}}{G}. \tag{3}$$

4. *The Galerkin Vector.* Since $\boldsymbol{S}$ is solenoidal, it may be represented by the curl of a vector function. Let

$$\boldsymbol{S} = -\text{curl } \boldsymbol{W}. \tag{4}$$

The functions ϕ and $\boldsymbol{S}$ are independent functions and we may therefore write

$$\phi = \frac{1}{\alpha}\,\text{div } \boldsymbol{W}. \tag{5}$$

Substituting equations (4) and (5) in the equilibrium equation (3), we obtain

$$\Delta[\text{grad div } \boldsymbol{W} - \text{curl curl } \boldsymbol{W}] = -\frac{\boldsymbol{K}}{G},$$

and, making use of the identity

$$\text{curl curl} = \text{grad div} - \Delta, \tag{6}$$

we obtain

$$\Delta\Delta\boldsymbol{W} = -\frac{\boldsymbol{K}}{G}. \tag{7}$$

Substituting equations (4) and (5) in equation (1), and employing the identity given in equation (6), we obtain

$$\rho = \Delta\boldsymbol{W} - \frac{1}{2(1-\mu)} \text{grad div } \boldsymbol{W}. \tag{8}$$

The vector function $\boldsymbol{W}$, satisfying equation (7) and producing the displacement according to equation (8), is identical with the Galerkin vector in the form given by Papkovitch. The relation between $\boldsymbol{W}$ and the Galerkin vector is

$$\boldsymbol{W} = \frac{1-\mu}{G}\boldsymbol{F}.$$

5. *The Papkovitch Functions.* The quantity in parentheses in equation (3) represents a vector function, say $\boldsymbol{B}$. Hence

$$\boldsymbol{B} = \alpha \text{ grad } \phi + \text{curl } \boldsymbol{S}, \tag{9}$$

and

$$\text{div } \boldsymbol{B} = \alpha\Delta\phi. \tag{10}$$

If the body force is taken equal to zero, equations (3) and (9) specify that $\boldsymbol{B}$ is harmonic, so that the complete solution of equation (10) may be written as

$$\phi = \frac{1}{2\alpha}(\boldsymbol{R}\cdot\boldsymbol{B} + \beta), \tag{11}$$

where $\boldsymbol{R}$ is the position vector of a field point referred to the origin and

$$\Delta \boldsymbol{B} = 0, \qquad \Delta\beta = 0. \tag{12}$$

Equations (1), (9), and (11) yield the expression for the displacement:

$$\boldsymbol{\varrho} = \boldsymbol{B} - \frac{1}{4(1-\mu)} \operatorname{grad} (\boldsymbol{R}\cdot\boldsymbol{B} + \beta). \tag{13}$$

The functions $\boldsymbol{B}$ and β, satisfying equations (12) and producing the displacement according to equation (13), are the Papkovitch functions. They bear the following relations to the corresponding functions $\phi_0, \phi_1, \phi_2, \phi_3$ found by Neuber:

$$\boldsymbol{B} = \frac{2(1-\mu)}{G} (i\phi_1 + j\phi_2 + k\phi_3),$$

$$\beta = \frac{2(1-\mu)}{G} \phi_0,$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit orthogonal vectors.

6. *Relations Among the Stress Functions.* When the body force is zero, the Galerkin, Papkovitch, and potential functions are found to be connected by the following relations. From equations (4), (5), and (9), we have

$$\begin{aligned} \boldsymbol{B} &= \Delta \boldsymbol{W} \\ &= \alpha \operatorname{grad} \phi + \operatorname{curl} \boldsymbol{S}; \end{aligned} \tag{14}$$

from equations (5), (11), and (14), we find

$$\begin{aligned} \beta &= 2 \operatorname{div} \boldsymbol{W} - \boldsymbol{R}\cdot\Delta\boldsymbol{W} \\ &= \alpha\left(2\phi - \boldsymbol{R}\cdot\operatorname{grad}\phi - \frac{1}{\alpha}\,\boldsymbol{R}\cdot\operatorname{curl}\boldsymbol{S}\right), \end{aligned} \tag{15}$$

and we have already seen that

$$\boldsymbol{S} = -\operatorname{curl} \boldsymbol{W},$$

$$\phi = \frac{1}{\alpha} \operatorname{div} \boldsymbol{W} = \frac{1}{\alpha} (\boldsymbol{R}\cdot\boldsymbol{B} + \beta).$$

COLUMBIA UNIVERSITY

THE FORM $wx+xy+yz+zu$

BY E. T. BELL

1. *Introduction.* In the usual notation,

$$N \equiv N[n = wx + xy + yz + zu; \quad w, x, z, u > 0; \quad y \geqq 0]$$

denotes the number of sets (w, x, y, z, u) of integers, subject to the conditions indicated, satisfying the stated equation in which n is an arbitrary constant integer >0. Let $\zeta_r(n)$ denote the sum of the rth powers of all the divisors of n, so that $\zeta_0(n)$ is the number of divisors. Then

$$N = \zeta_2(n) - n\zeta_0(n). \tag{1}$$

This curious result is the only one of the numerous theorems on quadratic forms stated by Liouville for which (apparently) no proof has been published.*

We shall first show that (1) follows from

$$2N' = \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n), \tag{2}$$

$$N' \equiv N'[n = wx + xy + yz + zu + ux; \quad w, x, y, z > 0; \quad u \geqq 0],$$

and then prove (2). Another similar result is stated in §5.

2. *Equivalence of* (1) *and* (2). The form in N' may be written

$$yz + (z + x)u + x(w + y);$$

and hence, by the conditions on the variables, $w+y \equiv y' > y$. Thus (2) is equivalent to

$$\begin{aligned} &\zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n) \\ &\quad = 2N'[n = yz + (z+x)u + xw; \quad x, y, z, w > 0; \quad u \geqq 0; \quad w > y]. \end{aligned} \tag{3}$$

Applying the substitution $(xz)(yw)$ to the last we see that (3) holds also when the condition $w>y$ is replaced by $w<y$.

Consider now the remaining possibility, $w=y$. The equation becomes

* J. Liouville, Comptes Rendus, Paris, vol. 62 (1866), p. 714; also, Journal de Mathématiques, (2), vol. 12 (1867), pp. 47–48. Noted in Dickson's *History*, vol. 3, p. 237. Liouville points out why the theorem is unusual.

$$(4) \qquad n = (x+z)(y+u); \qquad x, y, z > 0; \qquad u \geqq 0.$$

Hence if $n = d\delta$ is any resolution of n into a pair of positive divisors, the number of solutions of (4) for a fixed (d, δ) is

$$N[d = x+z;\ x, z > 0] \times N[\delta = y+u;\ y > 0;\ u \geqq 0],$$

that is, $(d-1)\delta$; and therefore the total number of solutions of (4) is $\sum(d-1)\delta$, the sum referring to all pairs (d, δ). Thus (4) has precisely $n\zeta_0(n) - \zeta_1(n)$ solutions.

But all the solutions of

$$(5) \qquad n = yz + (z+x)u + xw; \qquad x, y, z, w > 0; \qquad u \geqq 0$$

are exhausted by the three mutually exclusive sets in which $w > y$, $w < y$, $w = y$, respectively, and the number of solutions in each of these has just been determined. The total number of solutions of (5) is thus

$$\zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n) + n\zeta_0(n) - \zeta_1(n),$$

or $\zeta_2(n) - n\zeta_0(n)$. Hence (1) follows from (2). Conversely, by reversing the steps, (2) follows from (1), so that (1), (2) are equivalent.

3. *Dependence of* (2) *on an Auxiliary Relation* (6). Let $\phi(u, v)$ be finite and single-valued for all integer values of the variables u, v, and beyond the condition $\phi(u, v) = -\phi(v, u)$ for integer values of u, v, let $\phi(u, v)$ be arbitrary. Then

$$(6) \qquad \sum \phi(w+y, z) = \sum\left[\sum_{r=1}^{\delta-1} \phi(r, d)\right],$$

$\sum$ on the left referring to all (w, y, z), that on the right to all (d, δ), from

$$(7) \qquad d\delta = n = wx + xy + yz; \qquad d, \delta, w, x, y, z > 0,$$

in which all the letters denote integers and n is constant. Assuming this for the moment, we shall prove (2).

The form in (7) is invariant under the substitution $(xy)(zw)$. Hence $\sum w = \sum z$, the sums extending over all solutions of (7). Taking $\phi(u, v) \equiv u - v$ in (6), we get

$$(8) \qquad 2\sum y = \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n),$$

the left member of which is

$$2\sum N[y = y_1 + y_2;\quad y_1 > 0;\quad y_2 \geqq 0].$$

It follows that

$$2N[n = wx + (x + z)(y_1 + y_2);\quad w, x, z, y_1 > 0;\quad y_2 \geqq 0]$$

is given by the right member of (8). By a change of notation for the variables this result is (2).

4. *Proof of* (6). We now prove (6). The functions $h(u), f(u, v)$ are single-valued and finite for integer values of the variables, and beyond the conditions (for integer values of u, or of u, v)

$$h(u) = h(-u), \qquad f(u, v) = f(-u, v) = f(u, -v),$$

are arbitrary. Hence in a theorem concerning $f(u, v)$ we may replace $f(u, v) - f(v, u)$ by $\phi(h(u), h(v))$, where ϕ is as in §3. For f we have the identity*

$$\sum [f(d_1 + d_2, \delta_1 - \delta_2) - f(\delta_1 - \delta_2, d_1 + d_2)]$$
$$= \sum\left[(d-1)\{f(d,0) - f(0,d)\} + 2\sum_{r=1}^{d-1}\{f(r,d) - f(d,r)\}\right],$$

the sum on the right referring to all (d, δ), that on the left to all $(d_1, \delta_1, d_2, \delta_2)$, such that

$$d\delta = n = d_1\delta_1 + d_2\delta_2, \tag{9}$$

in which all letters denote integers >0 and n is constant. Hence

$$\sum \phi(h(d_1 + d_2), h(\delta_1 - \delta_2))$$
$$= \sum\left[(d-1)\phi(h(d), h(0)) + 2\sum_{r=1}^{d-1}\phi(h(r), h(d))\right]. \tag{10}$$

In (10) take $h(u) \equiv |u|$. Then

$$\sum \phi(d_1 + d_2, |\delta_1 - \delta_2|)$$
$$= \sum\left[(d-1)\phi(d,0) + 2\sum_{r=1}^{d-1}\phi(r,d)\right]. \tag{11}$$

* Equivalent to one stated by Liouville, Journal de Mathématiques, (2), vol. 3 (1858), pp. 282–284. The first proof, by elementary means, was given by T. Pepin, ibid., (4), vol. 4 (1888), pp. 84–92; I showed that the identity is equivalent to one between doubly periodic functions of the first and second kinds (Transactions of this Society, vol. 22 (1921), p. 215).

According as $\delta_1-\delta_2>0$, $\delta_1-\delta_2<0$, $\delta_1-\delta_2=0$ we have

$$\delta_1 = \delta_2 + \delta_1',\ \delta_1' > 0, \qquad n = d_1(\delta_2 + \delta_1') + d_2\delta_2;$$
$$\delta_2 = \delta_1 + \delta_2',\ \delta_2' > 0, \qquad n = d_1\delta_1 + d_2(\delta_1 + \delta_2');$$
$$n = \delta_1(d_1+d_2).$$

The third of these contributes $\sum(d-1)\phi(d, 0)$, summed over all divisors d of n, to the left of (11). The forms in the first two are equivalent under the substitution $(d_1d_2)(\delta_1\delta_2)(\delta_1'\ \delta_2')$; for the first, $|\delta_1-\delta_2| = \delta_1'$, for the second $|\delta_1-\delta_2| = \delta_2'$. Hence, by the equivalence just noted, these two together contribute $2\sum\phi(d_1+d_2,\ \delta_1')$ to the left of (11). Substituting these results into (9), (11), and changing the notation,

$$(d_2,\ \delta_2,\ d_1,\ \delta_1') = (w,\ x,\ y,\ z),$$

we get (6).

5. *Another Similar Result.* Other choices of ϕ in (6) give theorems on numbers of representations. From the results already given it is easily seen that

$$N[n = wx + xy + yz + zu + ux;\ x, y > 0;\ u, z, w \geqq 0] = \zeta_2(n).$$

To prove this we require

$$2N[n = x(w + y + u);\quad x, y, w > 0;\quad u \geqq 0] = \zeta_2(n) - \zeta_1(n),$$

which follows at once on noting that $x=d$, $w+y+u=\delta$, where $n=d\delta$, and that

$$N[\delta = w + y + u;\quad w, y > 0;\quad u \geqq 0]$$

is the coefficient of q^δ in the expansion of $q^2(1-q)^{-3}$, and hence is $\delta(\delta-1)/2$.

CALIFORNIA INSTITUTE OF TECHNOLOGY

CORRECTIONS TO MY PREVIOUS PAPER "TOPICS IN THE FUNCTIONAL CALCULUS"*

BY L. M. GRAVES

Professor Chittenden has called to my attention that on the first page of my paper I quoted an erroneous statement from his memoir *On general topology* · · · .† This statement was to the effect that the set-function H defined by him on page 298 is always additive, that is, $H(D+E)=H(D)+H(E)$ for every pair of sets D and E. Chittenden has communicated to me the following example in which the desired property fails to hold. Let limiting relations on the axis of reals be defined as usual, except that the origin belongs to the derivative $K(E)$ of a set E *only* in case it is both a limit point in the ordinary sense of the subset of E to the left of itself, and also a limit point of the subset of E to the right of itself. If the original set-function K is defined in this way, then the related function H is identical with K, and so fails to be additive.

However, if the set function H is defined in terms of a set-function K which is additive, it is easily shown that H will also be additive. This was the case on page 642 of my paper, where a second reference to the process of Chittenden occurs. Moreover, if K is an arbitrary set-function, a related function L may be defined which is monotone, and then the function L_a defined by the equation

$$L_a(E) = L(E) + \sum_D |L(E + D) - L(D)|$$

is always additive. From this it follows that the error does not essentially affect the statements made in my paper.

A second error in the paper (this one is not plagiarized) occurs at the bottom of page 642. The last seven lines on page 642 and the first three lines on page 643 should be deleted, as the three statements contained in these ten lines are all erroneous. The exact nature of the function $H(E)$ defined by Chittenden, when

* This Bulletin, vol. 41 (1935), p. 641.

† Transactions of this Society, vol. 31 (1929), p. 290.

applied to the example discussed in these lines, is unknown. In fact it is not clear that there exist sets E for which $H(E)$ is not vacuous. But it is obvious that $H(E)$ is always contained in $K(E)$, where K is the set-function (assumed additive) in terms of which H is defined.

In the example in question it is true that an "accessible" topology can be defined in terms of neighborhoods in such a way that the function Lx_n defined in terms of these neighborhoods is identical with the original function L, and so that the set of all continuous functions is dense on the whole space.

INSTITUTE FOR ADVANCED STUDY

ON (2, 2) PLANAR CORRESPONDENCES

BY L. H. CHAMBERS

1. *Introduction.* Most of the existing literature dealing with (2, 2) planar transformation is of the type given by the product of two harmonic homologies. By this I mean that the pairs of points of the plane π (or π'') are in harmonic homology. Papers of this type were given by E. Amson,* T. Kubota,† and P. Visalli.‡ Barraco§ defined an involutorial (2, 2) transformation of the plane by means of an involution between the tangents to a conic from points of the plane.

In this paper I shall consider only periodic (2, 2) transformations of period two. The treatment in each case, except those involving the Bertini involution, will be analytic. A synthetic treatment of some of the cases has been given by Sharpe and Snyder.‖ I shall use the following theorems proved in their paper.

A necessary and sufficient condition that the two images of a point P describe distinct loci as P moves on a curve C is that C touches the branch curve at every non-fundamental point they have in common.

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* Erlangen Dissertations, vol. 130 (1903–04).

† Science Reports, Tôhoku, vol. 6 (1918), and vol. 14 (1925).

‡ Circolo Matematico di Palermo, Rendiconti, vol. 3 (1889), pp. 165.

§ Giornale di Matematiche, vols. 53–54 (1915–16).

‖ Transactions of this Society, vol. 18 (1918), pp. 409.

A necessary and sufficient condition that a (2, 2) transformation be the product of a (2, 1) transformation and a (1, 2) transformation is that the defining curves of the one plane (and hence of the other) define a net.

Bertini proved that every rational involution of the plane was one of four types, namely, harmonic homology, Geiser, Jonquière, or Bertini.* Castelnuovo showed† that these four involutions could be mapped on a double plane and that planar involutions of any order are always rational.

I shall define the (2, 2) transformations of this paper as follows. Consider any of the transformations (H), (G), (J), (B) as existing in the planes π and π'', and mapped doubly upon a plane π' in such a manner that the two points in involution correspond to a single point of the plane π'. Associating points of π' with pairs of points in involution of the planes π and π'' will define a (2, 2) transformation.

The (2, 2) transformation, as defined, is periodic and of period two. A point P_1 of the plane π has for its image in π' a point P_1', and P_1' has for its image in π'' two points P_1'', P_2'', which are in involution. By the inverse transformation the points P_1'', P_2'', have for image the point P_1' of π'. The image of P_1' in π is the point P_1, and a point P_2, associated with P_1 in the involution of the plane π.

By this method of generating (2, 2) transformations, there are sixteen types. Of these, only ten are distinct. Various cases of each type arise due to the mapping used in the plane π'. Superposition of fundamental elements of the plane π' cause a reduction in the order of the transformations.

Zeuthen's theorem‡ will not apply in the Bertini types if the image curve, in the plane of the Bertini, degenerates unless both components of the curve are considered simultaneously. The mapping of (H), (G), and (J) upon a double plane has been done by Snyder§ and will not be repeated.

2. *The Mapping of* (B) *Upon a Double Plane.* This will be done by a different method than previously employed. The (B)

* Annali di Matematica, (2), vol. 8 (1877), p. 244.

† Rendiconti dei Lincei, (5), vol. 2 (1893), p. 205.

‡ F. Severi, *Trattato di Geometria Algebraico*, vol. 1, part 1, pp. 209.

§ V. Snyder, this Bulletin, vol. 30, pp. 101–124 (1920).

involution of π can be defined by the web of curves

$$a\phi^2 + b\phi\psi + c\psi^2 + df = 0, \tag{1}$$

where ϕ and ψ are general cubic curves and f is a sextic having double points at eight of the nine intersections of ϕ and ψ.*

Refer the points of π to those of a 3-way space by the equations

$$\xi = \phi^2, \quad \eta = \phi\psi, \quad \zeta = \psi^2, \quad \tau = f. \tag{2}$$

The pairs of points in involution are mapped doubly upon the cone $\Gamma \equiv \eta^2 - \xi\zeta = 0$ whose vertex arises from the ninth point of intersection of ϕ and ψ and whose generators correspond to cubics of the pencil $\Lambda \equiv \phi + \lambda\psi = 0$. By a stereographic projection† of Γ upon π', (B) will be mapped doubly upon π'.

The inverse transformation is obtained by solving the equations

$$\begin{aligned} \psi\xi - \phi\eta &= 0, \\ f\zeta - \psi^2\tau &= 0. \end{aligned} \tag{3}$$

Since sixteen roots of this solution are known, the resulting equation is quadratic. The coincidence curve is

$$K_9 \equiv \frac{\partial(\phi, \psi, f)}{\partial(x_1, x_2, x_3)} = 0. \tag{4}$$

We have $K_9 : 8Q_i^2$. A cubic of Λ meets K_9 in three variable points, hence K_9 is represented in the 3-way space as the intersection of Γ with a cubic surface and projects into $L_6' : (P_1' \equiv P_2')^3$ (that is, two consecutive 3-fold points). The tangent γ' to L_6' at $P_1' \equiv P_2'$ is determined by the tangent plane to Γ at the vertex of projection.

A line $l(x)$ is met by any cubic Λ in three points and is represented in π' by a $C_6' : (P_1' \equiv P_2')^3$, $4P'^2$. $8Q_i \sim 8C_2' : (P_1' \equiv P_2')$ with γ' as tangent. Points of γ' correspond to directions through P_1, P_2, the images of the vertex of projection. The line $l'(x')$ meets each fundamental conic in two points and has for image a $C_6 : 8Q_i^2$, P_1, P_2. A line $l'(x') : (P_1' \equiv P_2')$ will determine two generators of Γ, one of which is fixed; hence $(P_1' \equiv P_2') \sim C_3$ of Λ.

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* V. Snyder, American Journal of Mathematics, vol. 33 (1910), p. 43.

† Snyder and Sisam, *Analytic Geometry of Space*, p. 145.

3. *The Transformation* $(H)(H)$. Let the two transformations (H), of the planes π and π'', be mapped independently upon the plane π'.

CASE I. If the transformation (x') into $(\bar{x}')$ is

$$x_i' = \sum_{j=1}^{3} g_{ij}\bar{x}_j',$$

the (2, 2) transformation is obtained by combining the two (2, 1) transformations and this (1, 1) transformation. The F-points of π are (0, 1, 0) and $(\pm(G_{11}G_{31})^{1/2}, G_{21}, G_{31})$, which $\sim g_{31}x_1''^2 + g_{32}x_2''x_3'' + g_{33}x_3''^2 = 0$ and $x_3'' = 0$, respectively. L_4 is a degenerate quartic composed of two conics. A similar F system and branch curve exist in π''. The line $l(x) \sim C_4' : [(0, 1, 0) \equiv (0, 1, 0)]^2$. The line $l''(x'')$ has a similar image in π.

CASE II. If the transformation existing between (x') and $(\bar{x}')$ is $x_1' = \bar{x}_1'$, the (2, 2) transformation is rational. There are no F-elements in either π or π'', and $L_1 \equiv x_1 = 0$, $L_1'' \equiv x_1'' = 0$. The line $l(x) \sim C_2''$ composed of two lines. The line $l''(x'')$ has a similar image in π.

4. *The Transformation* $(G)(H)$. Combining the mapping of (G) of π with the mapping of (H) of π'', the resulting (2, 2) transformation is of the type (G) (H). $7Q_i \sim 7C_2'$. $P_1, P_2 \sim x_3'' = 0$. (0, 1, 0) of $\pi'' \sim x_1C_2 - x_2C_3 = 0$. $L_6 \equiv (x_2C_3 - x_3C_2)(x_1C_2 - x_2C_3) = 0$. $L_8' : [(0, 1, 0) \equiv (0, 1, 0)]^4$. The line $l(x) \sim C_6'' : P_1''^2$, $P_2''^2$ $[(0, 1, 0) = (0, 1, 0)]^4$, $2P''^2$. The line $l''(x'') \sim C_6 : 7Q_i^2$.

5. *The Transformation* $(J)(H)$. By combining the mapping of (H) of π'' with the mapping of (J) of π, the (2, 2) transformation (J) (H) is obtained.

CASE I. Let the transformation of the plane π' be

$$x_i' = \sum_{j=1}^{3} g_{ij}\bar{x}_j.$$

(0, 0, 1) of $\pi \sim C_{2(m-1)}'' : [(0, 1, 0) \equiv (0, 1, 0)]^{m-1}$, $2P''^{m-2}$. P_1, $P_2 \sim x_3'' = 0$. $4(m-1)Q_i \sim 4(m-1)C_2'' : (0, 1, 0)$. P_3, $P_4 \sim x_3'' = 0$. (0, 1, 0) of $\pi'' \sim C_{m+1} : (0, 0, 1)^{m-1}$. P_1'', P_2'' $\sim M_1 - a_3M_2 = 0$. P_3'', $P_4'' \sim a_1x_1 + a_2x_2 = 0$. $L_{2(m+1)}$ degenerates into two parts of order $(m+1)$, each part having an $(m-1)$-fold point at (0, 0, 1). $L_{4m}'' : [(0, 1, 0) \equiv (0, 1, 0)]^{2m}$, $2P^{2(m-1)}$, $2P^2$.

The line $l(x) \sim C''_{2(m+1)} : [(0, 1, 0) \equiv (0, 1, 0)]^{m+1}, 2P^m$. The line $l''(x'') \sim C_{2(m+1)} : (0, 0, 1)^{2(m-1)}, 4(m-1)Q_i^2, P_1^2, P_2^2$.

CASE II. Let the transformation of the plane π' be $x_i' = \bar{x}_i'$. $4(m-1)Q_i \sim 4(m-1)C_2''$. P_1, $P_2 \sim C_2''$. $(0, 0, 1) \sim C''_{2(m-1)} : [(0, 0, 1) \equiv (0, 0, 1)]^{m-2}$, $[(0, 1, 0) \equiv (0, 1, 0)]^{m-1}$. P_3, $P_4 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim M_2 = 0$. $(0, 0, 1)$ of $\pi'' \sim M_1 - a_3 M_2 = 0$. P_1'', $P_2'' \sim a_1 x_1 + a_2 x_2 = 0$. $L_{2(m+1)} \equiv x_1(a_1 x_1 + a_2 x_2)(M_1 - a_3 M_2) M_2$. $L''_{4m} : [(0, 1, 0) \equiv (0, 1, 0)]^{2m}$, $[(0, 0, 1) \equiv (0, 0, 1)]^{2(m-1)}$. The line $l(x) \sim C''_{2(m+1)} : [(0, 1, 0) \equiv (0, 1, 0)]^{m+1}$, $[(0, 0, 1) \equiv (0, 0, 1)]^m$. The line $l''(x'') \sim C_{2(m+1)} : 4(m-1)Q_i^2, P_1^2, P_2^2, (0, 0, 1)^{2(m-1)}$.

6. *The Transformation* (B) (H). CASE I. Let (H) of π'' be mapped upon π' and (B) of π be mapped upon π', so that $P_1' \equiv P_2'$ is a general point (a', b', c'). Eight points $Q_i \sim 8C_4'' : [(0, 1, 0) \equiv (0, 1, 0)]^2$. P_1, $P_2 \sim C_2'' : (0, 1, 0)$. $P_3 P_4 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim C_6 : 8Q_i^2, P_1, P_2$. $L_{12} : C_6 \cdot \overline{C}_6$, each of the net (1). $L_{12}'' : 2(P_1'' \equiv P_2'')^2$, $[(0, 1, 0) \equiv (0, 1, 0)]^6$. The line $l(x) \sim C''_{12} : [(0, 1, 0) \equiv (0, 1, 0)]^6, 8P''^2, 2(P_1'' \equiv P_2'')^3$. The line $l''(x'') \sim C_{12} : 8Q_i^4, P_1^2, P_2^2$.

CASE II. Let (B) be mapped on π' so that $P_1' \equiv P_2' = (0, 1, 0)$, $\gamma' \not\equiv x_1' = 0$, $\gamma' \not\equiv x_3' = 0$. $8Q_i \sim 8C_4'' : (0, 1, 0)^3$. $P_1 P_2 \sim C_2''$. $P_3 P_4 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim C_3$ of Λ. $L_6 : 2C_3$ of Λ. L_{12}'' has 9 branches through $(0, 1, 0)$ by threes in three directions, $p'' = 10$. The line $l(x) \sim C_{12} : 8P''^2$ and the singularities of L_{12}''. The line $l''(x'') \sim C_{12} : 8Q_i^2, P_1^2, P_2^2$.

CASE III. Let (B) be mapped upon π' so that $P_1' \equiv P_2' = (0, 1, 0)$, $\gamma' \equiv x_3' = 0$. $8Q_i \sim 8C_4'' : (0, 1, 0)^3$ with $x_3'' = 0$ as tangent. P_1, $P_2 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim C_3$ of Λ. $L_6 : 2C_3$ of Λ. L_{12}'' has 9 branches through $(0, 1, 0)$ with $x_3'' = 0$ as tangent. The line $l(x) \sim C_{12}'' : 8P''^2$ and singularities of L_{12}''. The line $l''(x'') \sim C_{12} : 8Q_i^4, P_1^2, P_2^2$.

7. *The Transformation* (G) (G). Combining two mappings of (G) we obtain the transformation (G) (G). $7Q_i \sim 7C_3''$. $L_{12} : 7Q_i^4$. The line $l(x) \sim C_9'' : 7Q_i^3, 2P''^2$. Similar results hold for the plane π''.

8. *The Transformation* (G) (J). Let the transformation (J) of π'' be mapped upon π'. Combining this mapping with that of (G), we obtain the transformation (G) (J). $7Q_i \sim 7C''_{m+1} : P_1'', P_2''$, $4(m-1)Q_i''$, $(0, 0, 1)^{m-1}$. P_1, $P_2 \sim M_1 - a_3 M_2 = 0$. P_3, $P_4 \sim a_1 x_1''$

$+a_2x_2''=0$. $4(m-1)Q_i' \sim 4(m-1)C_3 : 7Q_i$. P_1'', P_2' $\sim C_3 : 7Q_i$. $(0, 0, 1)$ of $\pi'' \sim C_{3(m-1)} : 7Q_i^{m-1}$, $2P^{m-2}$. $L_{6m} : 2P^{2(m-1)}$, $2P^2$, $7Q_i^{2m}$. $L''_{4(m+1)} : 4(m-1)Q_i^{1/4}$, $P_1''^4 P_2''^4$, $(0, 0, 1)^{4(m-1)}$. The line $l(x) \sim C''_{3(m+1)} : 4(m-1)Q_i^3$, $P_1''^3$, $P_2''^3$, $(0, 0, 1)^{3(m-1)}$, $2P^2$. The line $l''(x'') \sim C_{3(m+1)} : 7Q_i^{m+1}$, $2P^m$.

9. *The Transformation* $(B)(G)$. Let the transformation (G) of π'' be mapped upon π'. Combining this mapping with that of §2, we have the transformation $(B)(G)$. $8Q_i \sim 8C_6' : 7Q_i''^2$. P_1, $P_2 \sim C_3'' : 7Q_i'$. $7Q_i' \sim 7C_6 : 8Q_i^2$, P_1, P_2. $L_{24} : 8Q_i^3$, P_1^4, P_2^4. $L_{18}'' : 2(P_1' \equiv P_2'')^3$, $7Q_i^6$. The line $l(x) \sim C_{18}'' : 8P''^2$ and singularities of L_{18}''. The line $l''(x'') \sim C_{18} : 2P^2$, $8Q_i^6$, P_1^3, P_2^3.

10. *The Transformation* $(J)(J)$. Suppose the equations for the mapping of (J) of π'' upon π' are

$$x_1' x_3' - x_3' x_1'' = 0, \qquad x_2' N_1'' - v' N_2'' = 0,$$

where N_1'', N_2'' are curves of degree n having an $(n-2)$-fold point at $(0, 1, 0)$ and $v' = \sum_1^3 b_i x_i'$. Combining this mapping with that of (J) of π', we have the transformation $(J)(J)$. $4(m-1)Q_i \sim 4(m-1)C''_{n+1} : 4(n-1)Q_i''$, P_1'', P_2'', $(0, 1, 0)^{n-1}$. P_1, $P_2 \sim C''_{n+1} : (0, 1, 0)^{n-1}$, $4(n-1)Q_i''$, P_1'', P_2''. $(0, 0, 1)$ of $\pi \sim C''_{(m-1)(n+1)} : 4(n-1)Q_i''^{m-1}$, $P_1''^{m-1}$, $P_2''^{m-1}$, $(0, 1, 0)^{(m-1)(n-1)}$, $2P''^{m-2}$. $2Q \sim N_1'' - b_2 N_2''$. $2Q \sim b_1 x_1' + b_3 x_3'' = 0$. Similar images exist in π for the F-system of π''. $L_{2n(m+1)} : 2P^{2(n-1)}$, $2P^2$, $4(m-1)Q_i^{2n}$, P_1^{2n}, P_2^{2n}, $(0, 0, 1)^{2n(m-1)}$. $L''_{2m(n+1)}$ is similar to $L_{2n(m+1)}$. The line $l(x) \sim C_{(m+1)(n+1)} : 2P''^m$, $4(n-1)Q_i''^{m+1}$, $P_1''^{m+1}$, $P_2''^{m+1}$, $(0, 1, 0)^{(n-1)(m+1)}$. A similar image exists for $l''(x'')$.

11. *The Transformation* $(B)(J)$. Case I. Let (B), of π, be mapped upon π' so that $P_1' \equiv P_2' = (a', b', c')$. $8Q_i \sim 8C_{2(m+1)} : P_1''^2 P_2''^2$, $4(m-1)Q_i'^2$, $(0, 0, 1)^{2(m-1)}$. P_1, $P_2 \sim C''_{m+1} : P_1''$, P_2'', $4(m-1)Q_i$, $(0, 0, 1)^{m-1}$. $2Q \sim M_1'' - a_3 M_2''$. $2Q \sim a_1 x_1'' + a_2 x_2''$. $4(m-1)Q_i'' \sim 4(m-1)C_6$ of (1). P_1'', $P_2'' \sim C_6$ of (1). $(0, 0, 1)$ of $\pi'' \sim C_{6(m-1)} : 8Q_i^{2(m-1)}$, P_1^{m-1}, P_2^{m-1}, $2P^{m-2}$. $2Q_i'' \sim C_3$ of Λ. $L_{12m} : 8Q_i^{4m}$, P_1^{2m}, P_2^{2m}, $2P^2$, $2P^{2(m-1)}$. $L''_{6(m+1)} : (0, 0, 1)^{6(m-1)}$, $P_1'^6$, $P_2''^6$, $4(m-1)Q_i'^6$, $2(P_3' \equiv P_4'')$. $l(x) \sim C''_{6(m+1)} : 8P''^2$ and the singularities of $L''_{6(m+1)}$. $l''(x'') \sim C_{6(m+1)} : 8Q_i^{2(m+1)}$, P_1^{m+1}, P_2^{m+1}, $2P^m$.

CASE II. Map (B) upon π' so that $P_1' \equiv P_2' = (0, 0, 1)$. $8Q_i \sim 8_{(m+2)}'' : P_1'', P_2'', 4(m-1)Q_i', (0, 0, 1)^m$. $P_1, P_2 \sim C_1'' : (0, 0, 1)$. $2Q \sim a_1x_1'' + a_2x_2''$. One C_3 of $\Lambda \sim M_1'' - a_3M_2''$. $4(m-1)Q_i'', P_1'', P_2'' \sim (4m-3)\ C_3$ of Λ. $(0, 0, 1)$ of $\pi'' \sim C_{3m} : 8Q_i^m$. $L_{3(m+1)} : 8Q_i^{2(m+1)}, P_1^2, P_2^2, 2P^2$. $L_{3(m+2)}'' : P_1'^3, P_2''^3, 4(m-1)Q_i''$, $(0, 0, 1)^{3m}, 2(P_3'' \equiv P_4'')^3$. $l(x) \sim C_{3(m+2)}'' : 8P''^2$ and the singularities of $L_{3(m+2)}''$. $l''(x'') \sim C_{3(m+2)} : 8Q_i^{m+2}, P_1, P_2$. The complete image must be used to apply Zeuthen's theorem.

12. *The Transformation* (B) (B). CASE I. Let the two mappings of (B) upon π' be such that no F-elements coincide. $8Q_i \sim 8C_{12}'' : 8Q_i''^4, P_1''^2, P_2''^2$. $P_1, P_2 \sim C_6'' : 8Q_i''^2, P_1'', P_2''$. $2Q \sim C_3''$ of Λ''. $L_{36} : 8Q_i^{12}, P_1^6, P_2^6, 2(P_3 \equiv P_4)^3$. A similar F-system and branch curve exist in π''. $l(x) \sim C_3'' : 8P''^2$ and the singularities of L_{36}''. $l''(x'') \sim C_{36}$ similar to C_{36}''.

CASE II. Let the two mappings of (B) be such that $(P_1' \equiv P_2') = (\bar{P}_1' \equiv \bar{P}_2')$, $\gamma' \neq \bar{\gamma}'$. $8Q_i \sim 8C_9'' : 8Q_i''^3, P_1'', P_2''$. $P_1P_2 \sim C_3''$ of Λ''. One C_3 of $\Lambda \sim C_3''$ of Λ''. $L_{27} : 8Q_i^9, P_1^3, P_2^3$. A similar F-system and branch curve exist in π''. $l(x) \sim C_{27}'' : 8P''^2, 8Q_i^9, P_1^3, P_2^3$. The complete image must be used to apply Zeuthen's theorem.

CASE III. Let the two mappings of (B) be such that $(P_1' \equiv P_2') = (\bar{P}_1' \equiv \bar{P}_2')$, $\gamma' = \bar{\gamma}'$. $8Q_i \sim 8C_6'' : 8Q_i''^2, P_1'', P_2''$. $P_1, P_2 \sim P_1'', P_2''$. $L_{18} : 8Q_i^6$. $L_{18}' : 8Q_i''^6$. A similar F-system exists in π''. $l(x) \sim C_{18}^2 : 8Q_i''^6, 8P^2$. $l''(x'') \sim C_{18} : 8Q_i^6, 8P^2$.

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ON THE MAGNITUDE OF THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

BY EMMA LEHMER

Until very recently all the results of the investigations into the magnitude of the coefficients of the cyclotomic polynomial

$$Q_n(x) = \prod_{\delta|n} (1 - x^{n/\delta})^{\mu(\delta)} \tag{1}$$

tended to show that these coefficients are very small indeed. In fact for $n<105$ all the coefficients are ± 1, and 0, and for $n<385$ they do not exceed 2 in absolute value.

In 1883 Migotti* showed that the coefficients of $Q_n(x)$ are all ± 1 or 0 for n a product of two primes, but noted that the coefficient of x^7 in $Q_{105}(x)$ is -2. In 1895 Bang† proved that no coefficient of $Q_n(x)$ for $n=pqr$, $(p<q<r$, odd primes), exceeds $p-1$.

Nothing further was done on the problem until 1931, when I. Schur gave a very ingenious proof of the following theorem.

SCHUR'S THEOREM. *There exist cyclotomic polynomials with coefficients arbitrarily large in absolute value.*

As this proof has not been published, it is given below.‡

PROOF. Let $n=p_1p_2\cdots p_t$, where t is odd and $p_1<p_2<\cdots<p_t$ are odd primes such that§ $p_1+p_2>p_t$. To prove the theorem it is sufficient to show that the coefficient of x^{p_t} in $Q_n(x)$ is $1-t$. This can be done by taking $Q_n(x)$ modulo x^{p_t+1}. We then get

$$\begin{aligned} Q_n(x) &\equiv \prod_{i=1}^{t} (1 - x^{p_i})/(1 - x) \\ &\equiv (1 + x + \cdots + x^{p_t-1})(1 - x^{p_1})(1 - x^{p_2}) \cdots (1 - x^{p_t-1}) \\ &\equiv (1 + x + \cdots + x^{p_t-1})(1 - x^{p_1} - x^{p_2} - \cdots - x^{p_t-1}) \\ &\qquad \pmod{x^{p_t+1}}. \end{aligned}$$

* Sitzungsberichte, Akademie der Wissenschaften, Wien. (math), (2), vol. 87 (1883), pp. 7–14.

† Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.

‡ This proof is essentially the one given by Schur in a letter to Landau.

§ Such a set of primes exists for every t.

Collecting the coefficient of x^{p_t} in this last expression we see that it is precisely $-(t-1)$, so that as t increases we can exhibit arbitrarily large negative coefficients of the cyclotomic polynomials, which proves the theorem.

The question now remains as to the boundedness of the coefficients of $Q_n(x)$ for a fixed t. We have already seen that for $t=1$ and 2 these coefficients are actually bounded. The case $t=3$ was discussed by Bungers* who proved the following theorem.

BUNGERS' THEOREM. *As n runs over all products of three distinct primes, the cyclotomic polynomials $Q_n(x)$ contain arbitrarily large coefficients, provided there exist infinitely many prime pairs.*

His proof depends on choosing three primes, two of which differ by 2, and in exhibiting a coefficient of $Q_{pqr}(x)$ equal to $(p+1)/2$. It is the purpose of this note to modify Bungers' proof so as to eliminate the unproved assumption of the existence of infinitely many prime pairs.

Let $n=pqr$, where $q=kp+2$, and $r=(mpq-1)/2$. For a given p such primes q and r can always be found by Dirichlet's Theorem. We proceed to show that the coefficient of x^h, where $h=(p-3)(qr+1)/2$ is $(p-1)/2$ and hence can be made arbitrarily large with p. From (1) with $n=pqr$, we have

$$\begin{aligned} Q_{pqr}(x) &= \frac{(x^{pqr}-1)(x^p-1)(x^q-1)(x^r-1)}{(x-1)(x^{pq}-1)(x^{pr}-1)(x^{qr}-1)} \\ &\equiv (1+x+\cdots+x^{p-1})(1-x^q-x^r+x^{q+r}) \\ &\quad \cdot \sum x^{\nu qr+\lambda pr+\mu pq} \pmod{x^{pqr}}. \end{aligned}$$

Since we are interested in the coefficient of x^h, the summation indices ν, λ, μ, satisfy the following inequalities:

$$(2) \qquad \nu qr \leqq h, \qquad \lambda pr \leqq h, \qquad \mu pq \leqq h.$$

We now consider the diophantine equation

$$(3) \quad \nu qr + \lambda pr + \mu pq + \omega + \epsilon q + \eta r = (p-3)(qr+1)/2 = h,$$

where $\omega < p$, and $\epsilon = 0$ or 1, $\eta = 0$ or 1.

The coefficient of x^h is now given by the number of solutions of (3) with $\epsilon = \eta$ minus the number of solutions of (3) with $\epsilon \neq \eta$.

* Göttingen Dissertation, 1934.

Taking (3) modulo p, q, and r we have, since $qr \equiv -1 \pmod{p}$,

$$\nu qr + \omega + \epsilon q + \eta r \equiv 0 \pmod{p},$$
$$\lambda pr + \omega + \eta r \equiv (p-3)/2 \pmod{q},$$
$$\mu pq + \omega + \epsilon q \equiv (p-3)/2 \pmod{r}.$$

Multiplying the last two congruences by k and m, respectively, and remembering that $kpr \equiv 1 \pmod{q}$, while $mpq \equiv 1 \pmod{r}$, also that $q \equiv 2 \pmod{p}$, and $r \equiv -1/2 \pmod{pq}$, we get

$$\omega \equiv \nu - 2\epsilon + \eta/2 \pmod{p}, \tag{4}$$
$$\lambda \equiv k((p-3)/2 - \omega + \eta/2) \pmod{q}, \tag{5}$$
$$\mu \equiv m((p-3)/2 - \omega - \epsilon q) \pmod{r}. \tag{6}$$

We shall now show that if $\epsilon = \eta = 0$, (3) has $(p-1)/2$ solutions, while in the other three cases (3) has no solutions.

If $\epsilon = \eta = 0$, (4) gives us $\omega \equiv \nu \pmod{p}$ and since both ω and ν are less than p, $\omega = \nu$. Equations (5) and (6) become in this case

$$\lambda \equiv k((p-3)/2 - \nu) \pmod{q},$$
$$\mu \equiv m((p-3)/2 - \nu) \pmod{r}.$$

Since $\nu \leqq (p-3)/2$, and $k(p-3)/2$ and λ are $< q$, while $m(p-3)/2$ and μ are $< r$, these congruences are actually equalities, and we have determined for each of the $(p-1)/2$ values of ν, corresponding values of λ and μ, which are such that $\lambda \leqq k(p-3)/2$, so that

$$\lambda pr \leqq kpr(p-3)/2 < qr(p-3)/2 < h,$$

and $\mu \leqq m(p-3)/2$, so that

$$\mu pq \leqq mpq(p-3)/2 \leqq (2r+1)(p-3)/2 < h,$$

so that all the variables are determined within the ranges (2), and hence in the case $\epsilon = \eta = 0$, (3) has $(p-1)/2$ solutions.

For $\epsilon = 1$, $\eta = 0$, (4) gives us $\omega \equiv \nu - 2 \pmod{p}$. Hence either $\omega = \nu - 2$, or $\omega = p-1$, or $p-2$. In the last two cases we can use (5) to get

$$\lambda \equiv k((p-3)/2 - \omega) \equiv -k(p \pm 1)/2 \pmod{q}.$$

That is,

$$\lambda = q - k(p \pm 1)/2 \geqq q - k(p-1)/2,$$

so that

$$\lambda pr \geqq pqr - kpr(p-1)/2 > pqr - qr(p-1)/2$$
$$= qr(p+1)/2 > h.$$

Hence for $\omega = p-1$ or $p-2$, (2) is violated for λpr, and there are no solutions. If $\omega = \nu - 2 \leqq (p-7)/2$, we use (6) and obtain

$$\mu \equiv m((p-3)/2 - \omega - q) \pmod{r},$$

or

$$\mu = r + m((p-3)/2 - \omega - q) \geqq r + m(2-q).$$

Hence

$$\mu pq \geqq pqr + (2r+1)(2-q)$$
$$= (qr+1)(p-2) + (4r - p - q + 4)$$
$$> (qr+1)(p-2) > h,$$

so that (2) is again violated and there are no solutions of (3) for $\epsilon = 1, \eta = 0$.

In the next case $\epsilon = 0, \eta = 1$, we get from (4) $\omega = \nu + (p+1)/2$, and putting this value for ω in (6), we have

$$\mu \equiv m((p-3)/2 - \nu - (p+1)/2) \pmod{r}.$$

Hence $\mu = r - m(\nu+2) \geqq r - m(p+1)/2$, so that

$$\mu pq \geqq pqr - (2r+1)(p+1)/2 > pqr - (2r+1)(q-1)/2$$
$$= (qr+1)(p-1) + (2r - q - 2p + 3)/2$$
$$> (qr+1)(p-1) > h.$$

Thus this case does not yield any further solutions. We have now shown that (3) has at least $(p-1)/2$ solutions, since the remaining case $\epsilon = \eta = 1$ would contribute positively, if at all. In fact, it can be shown by a similar reasoning that this case does not contribute any solutions, so that the coefficient of x^h is precisely $(p-1)/2$. However, in any case, the coefficient of x^h increases with p, so that we have proved the following theorem.

THEOREM. *As n runs over all products of three distinct primes, the cyclotomic polynomials $Q_n(x)$ contain arbitrarily large coefficients.*

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THE CONTINUOUS ITERATION OF REAL FUNCTIONS*

BY MORGAN WARD AND F. B. FULLER

1. *Continuous Iterations.* Let $E(x)$ be a real, continuous, steadily increasing function of x in the range $-\infty < a \leqq x < \infty$ such that

$$E(x) > x, \qquad (x \geqq a), \tag{1}$$

and let $E_1(x) = E(x)$, $E_2(x) = E(E_1(x))$, $\cdots$ denote its successive iterates. In a previous note in this Bulletin, referred to hereafter as Note, one of us† has developed a simple formula for continuously iterating the function $E(x)$. We propose here to determine *all* continuous iterations of $E(x)$ subject to a restriction to be explained presently.

By a continuous iteration of $E(x)$ we shall understand a real function $\Theta_y(x)$ of the two real variables x and y with the following two properties

$$\text{(i)} \qquad \Theta_0(x) = x, \qquad \Theta_1(x) = E(x), \quad (x \geqq a).$$

$$\text{(ii)} \qquad \Theta_{y+z}(x) = \Theta_y(\Theta_z(x)), \qquad (x \geqq a,\ y, z \geqq 0).$$

The restriction which we shall impose upon the functions $\Theta_y(x)$ is the following:

(iii) $\Theta_y(a)$ *is a steadily increasing continuous function of* y *in the range* $0 \leqq y \leqq 1$.

2. *Prior Investigations.* The continuous iteration of real functions was discussed in detail by A. A. Bennett.‡ So far as the authors are aware, other investigators have confined their attention to the continuous iteration of analytic functions.§ The functional equation (ii) was first considered by A. Korkine,‖ who

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* Presented to the Society, February 29, 1936.

† Ward, *Note on the iteration of functions of one variable*, this Bulletin, vol. 40 (1934), pp. 688–690.

‡ Annals of Mathematics, (2), vol. 17 (1916), pp. 23–69.

§ See the references in the Note.

‖ Bulletin des Sciences Mathématiques, (2), vol. 6 (1882), part 1, pp. 228–242.

proved formally a result equivalent to the first theorem of this paper, assuming that $\Theta_y(x)$ was differentiable with respect to y.

A complete discussion of the functional equation $E_n(x)=x$ with x, $E(x)$ real, n a positive integer, has been given by J. F. Ritt,* and W. Chayoth† has recently proved certain very general existence theorems on functional equations in the real domain.

3. THEOREM 1. *Any function* Θ *satisfying the conditions* (i), (ii), *and* (iii) *is continuous and steadily increasing in both* x *and* y. *Moreover for each such function* $\Theta=E_y(x)$ *there exists a unique, continuous, steadily increasing solution* $\psi=f(x)$ *of the functional equation*

$$\psi(x+1)=E(\psi(x)), \qquad \psi(0)=0 \tag{2}$$

such that‡

$$E_y(x)=f(f^{-1}(x)+y), \qquad (x\geqq 0,\ y\geqq 0). \tag{3}$$

We have taken here and throughout the remainder of the paper, $a\neq 0$ and $E(a)=1$ as was shown to be possible without loss of generality in the Note.

To prove this theorem, let $\Theta=E_y(x)$ be a particular function satisfying the conditions (i), (ii), and (iii). Since $E_{x+1}(0)=E(E_x(0))$, we see from (iii) that $E_x(0)$ is continuous and steadily increasing in the range $0\leqq x<\infty$.

Write $f(x)$ for $E_x(0)$. Then $f(x)$ has a unique, continuous, steadily increasing inverse $f^{-1}(x)$ in the range $0\leqq x<\infty$ such that

$$f(f^{-1}(x))=f^{-1}(f(x))=x, \qquad f(0)=f^{-1}(0)=0.$$

Also $f(x+y)=E_{x+y}(0)=E_y(E_x(0))=E_y(f(x))$. Hence $f(x+1)=E(f(x))$, and $\psi=f(x)$ is a solution of (2). Then

$$E_y(x)=E_y(f\{f^{-1}(x)\})=f(f^{-1}(x)+y),$$

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* Annals of Mathematics, (2), vol. 17 (1916), pp. 113–122. See also the note by A. A. Bennett, loc. cit., p. 123.

† Monatshefte für Mathematik und Physik, vol. 39 (1932), pp. 279–288.

‡ The *converse* of this theorem is well known. See, for example, A. A. Bennett, Annals of Mathematics, volume cited, pp. 74–75; pp. 23–30.

which is (3). It is now evident that $E_y(x)$ is a continuous and steadily increasing function both of x and of y.

Finally, the function $f(x)$ in formula (3) is uniquely determined by $E_y(x)$. For letting $x=y=0$, and using (1), we see that $f(0)=0$. Hence $f^{-1}(0)=0$. Therefore on letting $x=0$ and $y=x$, $f(x)=E_x(0)$. The problem of determining all continuous iterations of $E(x)$ is thus reduced to the solution of the functional equation (2).

4. THEOREM 2. *Let $\theta(x)$ be a continuous function of x in the interval $0\leqq x<1$, which increases steadily from $\theta(0)=0$ to $\theta(1-0)=1$. Then every continuous steadily increasing solution ψ of the functional equation* (2) *is of the form*

$$\psi(x) = E_{[x]}(\theta(x - [x])), \tag{4}$$

where $[x]$ denotes the greatest integer in x.

Conversely, for every such choice of $\theta(x)$, (4) *gives a continuous steadily increasing solution of the functional equation* (2).

First of all, every such increasing solution ψ of (2) tends to infinity with x. For assume that $\psi(x)$ tends to a finite limit L as $x\to\infty$. Then $\psi(x)<L$ for all finite values of x. Now by (1), $E(L)>L$. Hence, since $E(x)$ is continuous, there exists a positive number δ such that $E(L-\delta)>L$. Choose x_0 so that $\psi(x)>L-\delta$, $x\geqq x_0$. Then $\psi(x+1)=E(\psi(x))>E(L-\delta)>L$, giving a contradiction.

It follows that in the interval $0\leqq x<\alpha$, $\psi(x)$ has a unique, continuous, steadily increasing inverse $\phi=\phi(x)=\psi^{-1}(x)$ such that $\phi\to\infty$ as $x\to\infty$. This inverse is readily seen to satisfy the famous functional equation of Abel,*

$$\phi(E(x)) = \phi(x) + 1, \qquad \phi(0) = 0. \tag{5}$$

For convenience, write e_n for $E_n(0)$, $(n=0, 1, 2, \cdots)$. Then $e_0=0$, $e_1=1$, and since by (1), $E(x)>x$, it follows that $e_n<e_{n+1}$.

We shall now show that $e_n\to\infty$. For otherwise, e_n tends to a finite limit k, and $e_n<k$, $(n=0, 1, 2, \cdots)$. Since $k>1$, if $E_{-1}(x)$ denotes the inverse of $E(x)$, then $E_{-1}(k)=M$, where $0<M<k$. For $k=E(E_{-1}(k))=E(M)>M$. Hence for all sufficiently large n, $e_n>M$. But then $e_{n+1}=E(e_n)>E(M)=k$, giving a contradiction.

* Works, vol. 2, Posthumous Papers, 1881, pp. 36–39.

It follows that, given any positive value of x, we can determine an integer k such that

$$e_k \leqq x < e_{k+1}. \tag{6}$$

Let x lie in the interval (6). Then from the properties of $E(x)$ and its ordinary iterates, we can write

$$x = E_k(y), \qquad y = E_{-k}(x), \qquad (0 \leqq y < 1),$$

where $E_{-k}(x)$ denotes the inverse of $E_k(x)$ in the interval $e_k \leqq x < \infty$.

Now in the interval $0 \leqq x < 1$, let us write $\theta^{-1}(x)$ for $\phi(x)$. Then $\theta^{-1}(0) = 0$, and $\theta^{-1}(x)$ increases steadily and continuously as x increases, and $\theta^{-1}(1-0) = \lim_{x=1}\theta^{-1}(x) = 1$ by (5). Furthermore, the inverse of $\theta^{-1}(x)$, which we denote by $\theta(x)$, exists and has the properties stated in Theorem 2. From (5), we see that

$$\phi(x) = \phi(E_k(y)) = \phi(y) + k = \theta^{-1}(y) + k$$

or

$$\phi(x) = \theta^{-1}(E_{-k}(x)) + k. \tag{7}$$

Since $0 \leqq E_{-k}(x) < 1$, we observe also that $k = [\phi]$, the greatest integer in $\phi(x)$.

To determine ψ, we need only solve (7) for x in terms of ϕ. We have

$$\phi - [\phi] = \theta^{-1}(E_{-[\phi]}(x)),$$

$$\theta(\phi - [\phi]) = E_{-[\phi]}(x),$$

$$x = E_{[\phi]}(\theta(\phi - [\phi])).$$

Hence*

$$\psi(x) = E_{[x]}(\theta(x - [x])). \tag{4}$$

The proof of the converse for a function θ satisfying the conditions of the theorem is almost word for word the same as in the special case $\theta(x) = x$, which has been given in full in the Note.

The function $\theta(x)$ is arbitrary save for the restrictions stated in the theorem. Once chosen, it fixes the iteration completely; it is in fact $E_x(0)$.

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* It is obvious that if $\pi(x)$ denotes a periodic function of x with period one, such that $\pi(x) = \theta(x)$, $(0 \leqq x < 1)$, then we can write $\psi(x) = E_{[x]}(\pi(x))$, or more concisely still, $\psi(x) = E_{[x]}(\psi(x - [x]))$, since $\psi(x - [x]) = \theta(x - [x])$.

ON POINCARÉ'S RECURRENCE THEOREM

BY CORNELIS VISSER

1. *Introduction.* Let S be a space in which is defined a measure μ such that $\mu(S)=1$. Suppose we are given a one parameter group of one to one transformations T_t, $(-\infty<t<\infty)$, of S into itself, with the properties:

(1) $$T_sT_t = T_{t+s}.$$

(2) For any measurable set E and any t the set T_tE is measurable and $\mu(T_tE)=\mu(E)$.

The following extension of Poincaré's recurrence theorem was proved by Khintchine.*

For any measurable E and any $\lambda<1$,

$$\mu(E\cdot T_tE) \geqq \lambda(\mu(E))^2$$

for a set of values t that is relatively dense on the t axis.

In this paper we give an elementary proof of this statement.

2. *An Auxiliary Theorem.* We prove the following theorem from which the recurrence theorem is an immediate consequence and which is also interesting in itself.

Let S be a space with a measure μ such that $\mu(S)=1$ and let $E_1, E_2, \cdots$ be an infinite sequence of measurable sets in S, all having a measure not less than m. Then for any $\lambda<1$ there exist in the sequence two sets E_i and E_k such that

$$\mu(E_iE_k) \geqq \lambda m^2.$$

Let us suppose that $\mu(E_iE_k)<p$ for any i and k. If we put

$$F_1 = E_1, \quad F_2 = E_2 - E_2F_1, \quad F_3 = E_3 - E_3F_2 - E_3F_1,$$
$$\cdots, \quad F_n = E_n - E_nF_{n-1} - \cdots - E_nF_1,$$

no two of the sets F have common points and F_i is part of E_i. Therefore

$$\mu(F_1) \geqq m, \quad \mu(F_2) > m - p, \quad \mu(F_3) > m - 2p,$$
$$\cdots, \quad \mu(F_n) > m - (n-1)p,$$

* A. Khintchine, *Eine Verschärfung des Poincaréschen "Wiederkehrsatzes,"* Compositio Mathematica, vol. 1 (1934), pp. 177–179.

and thus for $n=1, 2, \cdots$,

$$\mu(F_1 + \cdots + F_n) = \mu F_1 + \cdots + \mu F_n \geqq nm - \frac{1}{2} n(n-1)p.$$

It follows that

$$1 = \mu(S) \geqq \mu(F_1 + \cdots + F_n) \geqq nm - \frac{1}{2} n(n-1)p.$$

We now choose n such that

$$\frac{m}{p} < n \leqq \frac{m}{p} + 1.$$

Then we obtain

$$1 \geqq \frac{m}{p} m - \frac{1}{2}\left(\frac{m}{p} + 1\right)\frac{m}{p} p$$

$$= \frac{m^2}{2p} - \frac{m}{2},$$

or

$$p \geqq \frac{m^2}{2+m}.$$

Hence, if we exclude the trivial case $m=1$,

$$p > \frac{1}{3} m^2.$$

From this it follows that there must be two sets E_i and E_k with

$$\mu(E_i E_k) \geqq \frac{1}{3} m^2.$$

We shall now prove that the factor $1/3$ may be replaced by an arbitrary $\lambda < 1$. We consider the product space S^n, formed by the systems $(x_1, \cdots, x_n)$ of n points in S, and in this product space the sequence of sets $E_1^n, E_2^n, \cdots$. In S^n we can define a measure $\bar{\mu}$ such that the product of n measurable sets of S is measurable and has a measure that equals the product of the measures of its components. In applying the result we just ob-

tained to S^n and the sequence E_1^n, E_2^n, $\cdots$, we find two sets E_i^n and E_k^n such that

$$\mu(E_i^n E_k^n) \geqq \frac{1}{3}(m^n)^2.$$

Now

$$\mu(E_i^n E_k^n) = \mu((E_iE_k)^n) = (\mu(E_iE_k))^n,$$

and consequently

$$(\mu(E_iE_k))^n \geqq \frac{1}{3}(m^n)^2,$$

or

$$\mu(E_iE_k) \geqq \left(\frac{1}{3}\right)^{1/n} m^2.$$

Given $\lambda<1$, we can always define n such that $(1/3)^{1/n}\geqq\lambda$ and then select the pair E_i, E_k. This proves the theorem.

3. *Proof of the Recurrence Theorem.* Assume the contrary: There is a measurable set E and a number $\lambda<1$ such that

$$(*) \qquad \mu(E\cdot T_tE) < \lambda(\mu(E))^2$$

on arbitrarily large t-intervals. Let I_1 be a closed interval on which (*) holds; denote by $2l_1$ its length and by l_2 its center. There is an interval I_2 on which (*) holds and which has a length $>2(l_1+|l_2|)$. Denote by l_3 the center of I_2 and by I_3 an interval on which (*) holds and which has a length $>2(l_1+|l_2|+|l_3|)$, and so forth. Then the numbers l_k-l_i, $(i<k)$, belong to the intervals I_{k-i}; hence for any i and k, $(i<k)$,

$$\mu(E\cdot T_{l_k-l_i}E) < \lambda(\mu(E))^2,$$

and consequently

$$\mu(T_{l_i}E\cdot T_{l_k}E) < \lambda(\mu(E))^2$$

in contradiction to the theorem of §2, applied to the sequence $T_{l_1}E$, $T_{l_2}E$, $\cdots$. This proves the recurrence theorem.

It will be seen that it is not essential that t in T_t is a continuous parameter. The same method gives the same result in the case that t only runs through the values $0, \pm 1, \pm 2, \cdots$.

4. *Remark.* Let the sequence $E_1, E_2, \cdots$ be as in §2. Then we can even assert *that for every* $\lambda<1$ *there exists an infinite subsequence* $E_{i_1}, E_{i_2}, \cdots$ *such that for every* p *and* q

$$\mu(E_{i_p}E_{i_q}) \geqq \lambda m^2.$$

We show first that there exists an infinite subsequence $E_{k_1}, E_{k_2}, \cdots$ such that $\mu(E_{k_1}, E_{k_p}) \geqq \lambda m^2$ for every p. Suppose that no such subsequence exists; then to every $n=1, 2, \cdots$ belongs a p_n such that

$$\mu(E_nE_m) < \lambda m^2 \quad \text{for} \quad m \geqq n + p_n.$$

Writing $n_1=1$, $n_2=n_1+p_{n_1}$, $n_3=n_2+p_{n_2}, \cdots$, we have then for every i and k,

$$\mu(E_{n_i}E_{n_k}) < \lambda m^2,$$

which contradicts the theorem of §2. The proof is now easily completed by applying the diagonal principle.

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ON THE ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY MORRIS MARDEN

1. *Introduction.* The primary object of this note is to give a simple solution of a problem already discussed by many authors including the present one.† It is the problem of determining the regions within which lie the zeros of the derivative of a rational function when the zeros and poles of the function lie in prescribed circular regions.

THEOREM 1.‡ *For* $j=0, 1, \cdots, p$ *let* r_j *and* σ_j *be real constants*

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* Presented to the Society, September 4, 1934.

† For an expository account and list of references see M. Marden, American Mathematical Monthly, vol. 42 (1935), pp. 277–286, hereafter referred to as Marden I.

‡ See M. Marden, Transactions of this Society, vol. 32 (1930), pp. 81–109, hereafter referred to as Marden II.

with $\sigma_j^2 = 1$; let Z_j denote the circular region defined by the inequality

$$\sigma_j Z_j(z) \equiv \sigma_j(|z - \alpha_j|^2 - r_j^2) \leqq 0,$$

*and let z_j be an arbitrary point of the region Z_j. Then every zero of the derivative of the function**

$$f(z) = \prod_{j=0}^{p} (z - z_j)^{m_j}$$

satisfies at least one of the $p+2$ inequalities

$$\sigma_j Z_j(z) \leqq 0, \qquad (j = 0, 1, \cdots, p), \tag{1}$$

$$\frac{Z(z)}{\prod_{j=0}^{j=p} Z_j(z)} \equiv \left| \sum_{j=0}^{p} \frac{m_j(\bar{\alpha}_j - \bar{z})}{Z_j(z)} \right|^2 - \left(\sum_{j=0}^{p} \frac{|m_j| \sigma_j r_j}{Z_j(z)} \right)^2 \leqq 0. \tag{2}$$

Theorem 1 holds even when the m_j are complex numbers. When, however, they are all real, the use of the identities

$$(\alpha_j - z)(\bar{\alpha}_k - \bar{z}) + (\bar{\alpha}_j - \bar{z})(\alpha_k - z) = |\alpha_j - z|^2 + |\alpha_k - z|^2 - |\alpha_j - \alpha_k|^2,$$

and

$$|\alpha_j - z|^2 = Z_j(z) + r_j^2,$$

enables one, after expanding (2), to write

$$\frac{Z(z)}{\prod_{j=0}^{j=p} Z_j(z)} \equiv \sum_{j=0}^{p} \frac{n m_j}{Z_j(z)} - \sum_{j=0}^{p} \sum_{k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{Z_j(z) Z_k(z)}, \tag{3}$$

where $n = \sum m_j$ and

$$\tau_{jk} = |\alpha_j - \alpha_k|^2 - (|m_j| m_j^{-1} \sigma_j r_j - |m_k| m_k^{-1} \sigma_k r_k)^2.$$

The latter is the square of the common external or internal tangent of the circles $Z_j(z) = 0$ and $Z_k(z) = 0$ according as the product $|m_j m_k| (m_j m_k)^{-1} \sigma_j \sigma_k$ is positive or negative. The equation $Z(z) = 0$ represents for $n \neq 0$ a p-circular $2p$-ic and for $n = 0$ in general a $(p-1)$-circular $2(p-1)$-ic curve. The properties of these curves are studied in Marden II, p. 92.

* Where no limits are indicated, a product or summation is to be taken from $j = 0$ to $j = p$ and from $k = j+1$ to $k = p$.

2. *Three Lemmas.** (I). *If the points* $t_0, t_1, \cdots, t_p$ *varying independently of one another describe the closed interiors of the circles* $T_0, T_1, \cdots, T_p$, *respectively, the center and radius of* T_j *being* γ_j *and* ρ_j, *then the point* $w=\sum_{j=0}^{p} m_j t_j$ *describes the closed interior of a circle* W *with center at* $\gamma=\sum_{j=0}^{p} m_j\gamma_j$ *and radius of* $\rho=\sum_{j=0}^{p}|m_j|\rho_j$.

For

$$|w-\gamma| = \left|\sum_{j=0}^{p} m_j(t_j-\gamma_j)\right| \leqq \rho;$$

conversely, if k and θ are arbitrary, $0\leqq k\leqq 1$, and if $m_j(t_j-\gamma_j) = k|m_j|\rho_j e^{i\theta}$, then $w-\gamma=\sum k|m_j|\rho_j e^{i\theta}=k\rho e^{i\theta}$.

(II). *If the points* t_j, $(j\geqq 1)$, *vary as in Lemma* (I), *but the point* t_0 *describes the closed exterior of its circle* T_0, *the locus of the point* w *is the closed exterior of a circle with center at* $\gamma=\sum_{j=0}^{p} m_j\gamma_j$ *and radius* $\rho=2|m_0|\rho_0-\sum_{j=0}^{p}|m_j|\rho_j$ *provided* $\rho>0$, *and is the entire plane if* $\rho\leqq 0$.

For, when $\rho>0$,

$$|w-\gamma| \geqq |m_0(t_0-\gamma_0)| - \left|\sum_{j=1}^{p} m_j(t_j-\gamma_j)\right| \geqq \rho;$$

conversely, if k and θ are arbitrary with $k\geqq 1$, and if

$$m_0(t_0-\gamma_0) = [|m_0|\rho_0 + (k-1)\rho]e^{i\theta},$$

and

$$m_j(t_j-\gamma_j) = -|m_j|\rho_j e^{i\theta}, \qquad (j\geqq 1),$$

then $w-\gamma=k\rho e^{i\theta}$.

If ρ_0 decreases while ρ_j, $(j\geqq 1)$, remain constant, ρ will approach zero and the locus of w will become the entire plane. The locus is, therefore, the entire plane for $\rho\leqq 0$.

(III). *If the points* t_j, $(j>k\geqq 1)$, *vary as in Lemma* (I), *but the points* t_j, $(j\leqq k)$, *describe the exteriors of their circles* T_j, *the locus* W *of* w *is the entire plane.*

For, if each t_j, $(1\leqq j\leqq k)$, were to vary merely interior to a

* See J. L. Walsh, Transactions of this Society, vol. 24 (1922), p. 61 and p. 169; also H. Minkowski, Collected Works, vol. 2, p. 177.

circle T_j' drawn exterior to but not enclosing the circle T_j while the remaining t_j vary as indicated in Lemma (III), and if the radius ρ_j' of T_j' were chosen so that

$$|m_0|\rho_0 - \sum_{j=1}^{k}|m_j|\rho_j' - \sum_{j=k+1}^{p}|m_j|\rho_j = 0,$$

then by Lemma (II) the locus of w would be the entire plane.

3. *Proof of Theorem* 1. Let z be any fixed point exterior to all the regions Z_j; that is, let z be such that $\sigma_j Z_j(z) > 0$, all j. Since $\sigma_j Z_j(z_j) \leqq 0$, point $t_j = (z_j - z)^{-1}$ lies in or on the circle T_j with center $\gamma_j = (\bar{\alpha}_j - \bar{z})/Z_j(z)$ and radius $\rho_j = (\sigma_j\gamma_j)/Z_j(z)$. According to Lemma (I), the locus W_z of the point $w = \sum m_j t_j$ will be defined by the inequality:

$$(4) \qquad \left| w - \sum_{j=0}^{p}\frac{m_j(\bar{\alpha}_j - \bar{z})}{Z_j(z)}\right|^2 - \left(\sum_{j=0}^{p}\frac{|m_j|\sigma_j\gamma_j}{Z_j(z)}\right)^2 \leqq 0.$$

Now, in order to be a zero of $f'(z)$, point z must be a root of the equation

$$(5) \qquad -\frac{f'(z)}{f(z)} = \sum_{j=0}^{p}\frac{m_j}{z_j - z} = 0;$$

that is, point $w = 0$ must satisfy inequality (4). Hence, any zero of $f'(z)$, not satisfying any of the inequalities (1), must satisfy (2); that is to say, $\sigma Z(z) \leqq 0$, where $\sigma = \prod\sigma_j$.

4. *A Locus Problem.* What then is the locus Z of the zeros of $f'(z)/f(z)$ when the points z_j vary independently within their circular regions Z_j?

Theorem 1 reveals that $\sigma Z(z) \leqq 0$ for any point z of locus Z exterior to all the regions Z_j. Conversely, if exterior to all the regions Z_j, any point z for which $\sigma Z(z) \leqq 0$ belongs to the locus Z. With the aid of Lemma (II), it can be shown that a point z interior to just one region Z_j belongs to locus Z if and only if either $\sigma Z(z) \leqq 0$ or $\sigma S(z) \leqq 0$, where

$$\frac{S(z)}{\prod Z_j(z)} = \sum_{j=0}^{p}\frac{|m_j|r_j\sigma_j}{Z_j(z)}.$$

(The curve $S(z) = 0$, in general a p-circular $2p$-ic, consists only of points on the boundaries of two or more regions Z_j or interior

to at least one region Z_j, the points z interior to just one region Z_j satisfying inequality $\sigma Z(z) \leqq 0$.) Likewise, with the aid of Lemma (III), it can be shown that every point common to two or more regions Z_j belongs to locus Z.

If a given point z is to be on the boundary of locus Z, point $w=0$ must cease to be a point of W_z whenever the regions Z_j and hence W_z are diminished, no matter how slightly. That is to say, the point $w=0$ must be on the boundary of the locus W_z. This implies that $Z(z)=0$ if z, a boundary point of locus Z, is exterior to all the regions Z_j or interior to just one region Z_j with $\sigma Z(z) \leqq 0$. It also implies that no boundary point z of locus Z may be either interior to just one region Z_j with $\sigma S(z) \leqq 0$, or interior to two or more regions Z_j; for, in those cases, the locus W_z is the entire plane.

In short, the locus Z is a set of regions bounded by the ovals of the curve $Z(z)=0$, each region, according to Marden II, being simply-connected.

5. *Applications.* When $p=2$ and $n=m_0+m_1+m_2=0$, equation (5) may be written as the cross-ratio

$$(6) \qquad (z_0 z_1 z_2 z) \equiv \frac{(z_0 - z_2)(z_1 - z)}{(z_0 - z)(z_1 - z_2)} = -\frac{m_1}{m_0}.$$

Here

$$Z(z) \equiv -m_0 m_1 r_{01} Z_2(z) - m_1 m_2 r_{12} Z_0(z) - m_2 m_0 r_{20} Z_1(z) = 0,$$

$$S(z) \equiv \left| m_0 \right| r_0 \sigma_0 Z_1(z) Z_2(z) + \left| m_1 \right| r_1 \sigma_1 Z_2(z) Z_0(z) + \left| m_2 \right| r_2 \sigma_2 Z_0(z) Z_1(z) = 0,$$

represent in general a circle and bicircular quartic, respectively. If λ denotes the coefficient of the term (x^2+y^2) in the expression $Z(z)$, the region $\sigma Z(z) \leqq 0$ is the interior or exterior of the circle $Z(z)=0$ according as $\sigma\lambda>0$ or $\sigma\lambda<0$. Hence, if all the points for which $\sigma S(z) \leqq 0$ lie in the region $\sigma Z(z) \leqq 0$, the locus Z will be the interior or exterior of circle $Z(z)=0$ according as $\sigma\lambda>0$ or $\sigma\lambda<0$. If, however, not all the points for which $\sigma S(z) \leqq 0$ lie in the region $\sigma Z(z) \leqq 0$, the locus will be the entire plane. This discussion verifies the following theorem due to Walsh.*

* J. L. Walsh, Transactions of this Society, vol. 22 (1921), pp. 101–116, and Rendiconti di Palermo, vol. 46 (1922), pp. 1–13. See also A. B. Coble, this Bulletin, vol. 27 (1921), pp. 434–437; T. Nakahara, Tôhoku Mathematical Journal, vol. 23 (1924), p. 97; and Marden II.

If the points z_0, z_1, and z_2 varying independently of one another describe given circular regions Z_0, Z_1, and Z_2, then the point z defined by the constant cross-ratio $(z_0z_1z_2z) = c$ also describes a circular region Z.

On allowing a number of the regions Z_j to coincide, we deduce from Theorem 1 the following corollary.

COROLLARY 1. *If all the zeros of a polynomial $f_j(z)$ of degree n_j lie in the circular region Z_j, then every zero of the derivative of the product*

$$\prod_{j=0}^{p} [f_j(z)]^{q_j}$$

will satisfy at least one of the $p+2$ inequalities (1) *and* (2) *with $m_j = n_jq_j$.**

In particular, upon setting $f_j(z) = f(z) - \gamma_j$, we obtain from Corollary 1 a generalization of a theorem stated by Jentsch and proved by Fekete.†

COROLLARY 2. *If all the points at which a given polynomial $f(z)$ takes on the value γ_j lie in the circular region Z_j, then every root z of the equation*

$$\sum_{j=0}^{p} \frac{m_j}{f(z) - \gamma_j} = 0$$

satisfies at least one of the $p+2$ inequalities (1) *and* (2).

6. *Generalizations.* By requiring $w = \lambda$ instead of $w = 0$ to satisfy inequality (4), we are led to the following result.

THEOREM 2. *Under the hypotheses of Theorem* 1 *or of Corollary* 1, *every zero of the function $f'(z) + \lambda f(z)$ satisfies at least one of the $p+2$ inequalities*

$$\sigma_j Z_j(z) \leqq 0, \qquad (j = 0, 1, \cdots, p), \tag{7}$$

$$\left| \lambda - \sum_{j=0}^{p} \frac{m_j(\bar{\alpha}_j - \bar{z})}{Z_j(z)} \right|^2 - \left(\sum_{j=0}^{p} \frac{|m_j| \sigma_j r_j}{Z_j(z)} \right)^2 \leqq 0. \tag{8}$$

* This corollary contains as special cases a number of important theorems due to Gauss-Lucas, Laguerre, Bôcher, and Walsh. See Marden I.

† R. Jentsch, Archiv der Mathematik und Physik, vol. 25 (1917), p. 196, prob. 526; M. Fekete, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 31 (1922), pp. 42–48; Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 61, probs. 126–127.

If all the m_j are real, the left-hand side of (8) may be rewritten, with the aid of the identities given in §1, as

$$(9) \qquad \sum_{j=0}^{p} \frac{|\lambda|^2 m_j \Gamma_j(z)}{n Z_j(z)} - \sum_{j=0}^{p} \sum_{k=j+1}^{p} \frac{m_j m_k r_{jk}}{Z_j(z) Z_k(z)},$$

where

$$\Gamma_j(z) \equiv |z - (\alpha - n\lambda^{-1})|^2 - r_j^2.$$

The equation $\Gamma_j(z) = 0$ represents the circle obtained by translating the circle $Z_j(z) = 0$ in the direction and magnitude of the vector n/λ. Set equal to zero, expression (7) represents a $(p+1)$-circular $2(p+1)$-ic curve with singular foci at the roots of the equation

$$\lambda + \sum_{j=0}^{p} \frac{m_j}{z - \alpha_j} = 0.$$

In particular, assuming the hypotheses of Corollary 1, and setting $\sigma_0 - 1 = p = p_0 - 1 = 0$, we find this curve to reduce to the circle $\Gamma_0(z) = 0$. In other words, *if all zeros of a polynomial $f(z)$ of degree n lie in a given circle, any zero of the linear combination $f'(z) + \lambda f(z)$ will lie either in the given circle or in the one obtained by translating the given circle in the direction and magnitude of the vector $n\lambda^{-1}$.**

Finally, by requiring $w = g(z)$, an arbitrary function of z, instead of $w = 0$, to satisfy inequality (4), we obtain a theorem similar to Theorem 2 for the zeros of the function $f'(z) + g(z)f(z)$. For example, if $g(z) = \bar{z}$, and if all the zeros of a polynomial $f(z)$ of degree n lie in the circle $|z| \leqq r \leqq 2n^{1/2}$, all zeros of the function $\bar{z}f(z) + f'(z)$ lie in the same circle.

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* See M. Fujiwara, Tôhoku Mathematical Journal, vol. 9 (1916), pp. 102–108; T. Takagi, Proceedings of the Physico-Mathematical Society of Japan, vol. 3 (1921), pp. 175–179; J. L. Walsh, this Bulletin, vol. 30 (1924), p. 52.

ON CERTAIN DISTRIBUTION THEOREMS OF STATISTICS*

BY SOLOMON KULLBACK

1. *Introduction.* The problems of statistics have been classified under three heads: problems of specification, problems of estimation, problems of distribution.† I intend to prove certain theorems regarding the problem of distribution in more general form than that in which they have been proved before.

By the distribution problem of statistics is meant the problem of finding the distribution function, or law of total probability, of the real, single-valued functions $\mu_\alpha(x_1, x_2, \cdots, x_n)$, $(\alpha = 1, 2, \cdots, r)$, given the distribution function of the real variables $x_1, x_2, \cdots, x_n$. Many particular problems of this nature have been solved. More general solutions have been obtained, imposing certain restrictions on the functions $\mu_\alpha(x_1, x_2, \cdots, x_n)$ and the distribution function of $x_1, x_2, \cdots, x_n$.‡

2. *Distribution Functions.* By the distribution function, or law of total probability of the real variables $x_1, x_2, \cdots, x_n$, is meant the monotone, absolutely additive set function $\phi(E)$, where $0 \leqq \phi(E) \leqq 1$, and $\phi(S) = \int_S d_{x_1 x_2 \cdots x_n} \phi(E) = 1$, where S is the whole $x_1 x_2 \cdots x_n$ space.§

* Presented to the Society, April 19, 1935, under the title: *On the distribution problem of statistics.*

† R. A. Fisher, *On the mathematical foundations of theoretical statistics*, Philosophical Transactions of the Royal Society, vol. 222 A (1922), p. 309 ff.

‡ C. V. L. Charlier, *Some fundamental problems in the theory of probability*, Arkiv för Matematik, Astronomi och Fysik, vol. 8 (1912), No. 4, pp. 1–39; T. Kameda, I. *Eine Verallgemeinerung des Poissonschen Problems in der Wahrscheinlichkeitsrechnung*, Proceedings Physico-Mathematical Society, Tokyo, vol. 9 (1917–18), pp. 155 ff.; II. *Theory of generating functions and its application to the theory of probability*, Journal Faculty of Science, University of Tokyo, sec. I, vol. 1 (1925), part 1, pp. 1–62; E. L. Dodd, *The frequency law of a function of variables with given frequency laws*, Annals of Mathematics, (2), vol. 27 (1925), pp. 12–20; S. Kullback, *An application of characteristic functions to the distribution problem of statistics*, Annals of Mathematical Statistics, vol. 5 (1934), pp. 263–307.

§ See E. K. Haviland, I. *On the theory of absolutely additive distribution functions*, American Journal of Mathematics, vol. 56 (1934), pp. 625–658, particularly pp. 627–628.

3. *The Distribution Function of the Functions* $\mu_\alpha(x_1, x_2, \cdots, x_n)$. For convenience we shall consider the case of two functions; the problem for r functions is solved by the same methods.

Let $\mu_\alpha(x_1, x_2, \cdots, x_n)$, $(\alpha = 1, 2)$, be real, single-valued functions of the real variables $x_1, x_2, \cdots, x_n$, where the distribution function of the x_i, $(i = 1, 2, \cdots, n)$, is $\phi(E)$. Let the functions $\mu_\alpha(x_1, x_2, \cdots, x_n)$, $(\alpha = 1, 2)$, be measurable with respect to $\phi(E)$, that is, for every number A, the set of those points P in which $\mu_\alpha(P) > A$, belongs to the definition region T of ϕ.*

For convenience we shall write $\mu_\alpha(x)$ for $\mu_\alpha(x_1, x_2, \cdots, x_n)$, $t\mu(x)$ for $t_1\mu_1(x_1, x_2, \cdots, x_n) + t_2\mu_2(x_1, x_2, \cdots, x_n)$ and $d_x\phi(E)$ for $d_{x_1x_2\cdots x_n}\phi(E)$. Accordingly, since $e^{it\mu(x)}$ is measurable with respect to $\phi(E)$ and *eo ipso* summable with respect to $\phi(E)$,† the integral $\int_R e^{it\mu(x)} d_x\phi(E)$, where $i = \sqrt{(-1)}$, exists for all values of t_1 and t_2, and is Radon's generalization of the Lebesgue-Stieltjes integral.‡ Since $\phi(E)$ is of finite total variation, the limit of the above integral exists as $R \to S$, where R is an arbitrarily large rectangle and S is the entire $x_1x_2 \cdots x_n$-space, and we write

$$\Lambda(t_1, t_2; \mu_\alpha, \phi) = \int_S e^{it\mu(x)} d_x\phi(E). \tag{1}$$

We must now define the appropriate distribution function in the $\mu_1\mu_2$-space. Let E' be any set in the $\mu_1\mu_2$-space contained in an arbitrarily large rectangle, and let E be the corresponding set in the $x_1x_2 \cdots x_n$-space.

We then define the set function Φ by $\Phi(E') = \phi(E)$. Since $\phi(E)$ is a distribution function and $\mu_\alpha(x)$, $(\alpha = 1, 2)$, are single-valued functions of $x_1, x_2, \cdots, x_n$, it follows that $0 \leqq \Phi(E') \leqq 1$ and that Φ is absolutely additive. We shall show below that $\Phi(S') = 1$, where S' is the whole $\mu_1\mu_2$-space. Thus $\Phi(E')$ is the appropriate distribution function in the $\mu_1\mu_2$-space.

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* J. Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungsberichte der Mathematischen-Naturwissenschaftlichen Klasse der Kaiserlichen Akademie zu Wien, vol. 122 (1913), pp. 1295–1438, particularly pp. 1299 and 1325. E. Hopf, *On causality, statistics and probability*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 13 (1934), pp. 51–102, particularly §1. J. L. Doob, *Probability and statistics*, Transactions of this Society, vol. 36 (1934), pp. 759–775.

† J. Radon, loc. cit., p. 1326.

‡ J. Radon, loc. cit., pp. 1322–1332.

4. *The Characteristic Function of the Distribution Function of* $\mu_\alpha(x)$.* We shall now show that

$$\int_S e^{it\mu(x)} d_x\phi(E) = \int_{S'} e^{it\mu} d_\mu\Phi(E'), \tag{2}$$

where $t\mu \equiv t_1\mu_1 + t_2\mu_2$ and $d_\mu\Phi(E') \equiv d_{\mu_1\mu_2}\Phi(E')$.

Separate the real and imaginary parts of (2) and consider, for example, the approximating sums†

$$S_n = \sum_{k=-\infty}^{\infty} y_k^{(n)} \phi(E_k^{(n)}),$$

where $E_k^{(n)}$ is the set of points $(x_1, x_2, \cdots, x_n)$ for which

$$y_k^{(n)} \leqq \cos(t\mu(x)) < y_{k+1}^{(n)}, \quad \text{and} \quad S_n' = \sum_{k=-\infty}^{\infty} y_k^{(n)} \Phi(E_k'^{(n)}),$$

where $E_k'^{(n)}$ is the set of points (μ_1, μ_2) for which $y_k^{(n)} \leqq \cos(t\mu) < y_{k+1}^{(n)}$. The set $E_k^{(n)}$ belongs to the domain of definition of ϕ and $E_k'^{(n)}$ to that of Φ. The equality of the approximating sums follows from the fact that in accordance with the definition in §3, $\phi(E_k^{(n)}) = \Phi(E_k'^{(n)})$. A similar result follows for the imaginary part, and thus (2) is established.

If in (2) we set $t_1 = t_2 = 0$, we have

$$\int_{S'} d_\mu\Phi(E') = \int_S d_x\phi(E) = 1,$$

that is, $\Phi(S') = 1$, as was mentioned above.

5. *Application of an Inversion Formula.* By E. K. Haviland's inversion formula for Fourier-Stieltjes transforms,‡ we have

$$g(\nu_1, \nu_2) = \frac{1}{(2\pi)^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\right) \frac{(e^{-it_1\xi_1} - 1)(e^{-it_2\xi_2} - 1)}{(it_1)(it_2)} e^{-it_1\nu_1 - it_2\nu_2} \cdot \Lambda(t_1, t_2; \Phi) dt_1 dt_2, \tag{3}$$

where

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* P. Lévy, *Calcul des Probabilités*, 1925, p. 161.

† J. Radon, loc. cit., pp. 1326–1328.

‡ E. K. Haviland, II. *On the inversion formula for Fourier-Stieltjes transforms in more than one dimension*, American Journal of Mathematics, vol. 57 (1935), pp. 94–100.

$$g(\nu_1, \nu_2) = \Phi(Q + P_{\nu_1\nu_2}) = G(\nu_1 + \xi_1, \nu_2 + \xi_2) - G(\nu_1 + \xi_1, \nu_2) - G(\nu_1, \nu_2 + \xi_2) + G(\nu_1, \nu_2),$$

where Q is $(0 \leqq \omega_1 < \xi_1; 0 \leqq \omega_2 < \xi_2)$, and $Q + P_{\nu_1\nu_2}$ is a non-singular rectangle* of Φ, where also $G(\mu_1, \mu_2)$ is the point function corresponding to $\Phi(E')$,

$$\Lambda(t_1, t_2; \Phi) = \int_{E'} e^{it_\mu} d_\mu \Phi(E'),$$

and the double integral in (3) is a Cauchy principal value, as the parentheses indicate. We thus have the following theorem.

THEOREM. *If $\mu_\alpha(x)$, $(\alpha = 1, 2)$, are real, single-valued functions which are measurable with respect to the monotone, absolutely additive set function $\phi(E)$ which is the distribution function of the real variables $x_1, x_2, \cdots, x_n$, then* (a) *the characteristic function of the distribution function of μ_1, μ_2 is given by*

$$\Lambda(t_1, t_2; \mu_\alpha, \phi) = \int_E e^{it_{\mu(x)}} d_x \phi(E),$$

and (b) *the distribution function of μ_1, μ_2 is essentially† determined by setting $\Phi(Q + P_{\nu_1\nu_2})$ equal to the expression on the right-hand side of* (3), *with $\Lambda(t_1, t_2; \mu_\alpha, \phi)$ in place of $\Lambda(t_1, t_2; \Phi)$.*

If $\phi(E)$ is absolutely continuous and $\Lambda(t_1, t_2; \mu_\alpha, \phi)$ is absolutely integrable, then‡

$$h(\mu) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_\mu} \Lambda(t_1, t_2) dt_1 dt_2, \tag{4}$$

where

$$\Lambda(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_{\mu(x)}} f(x) dx, \qquad dx = dx_1 dx_2 \cdots dx_n,$$

and where $f(x) \equiv f(x_1, x_2, \cdots, x_n)$ and $h(\mu) \equiv h(\mu_1, \mu_2)$ are the respective probability densities in the $x_1\, x_2 \cdots x_n$ space and $\mu_1\mu_2$-space.§

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* E. K. Haviland, loc. cit., I, p. 627–628.

† See E. K. Haviland, loc. cit., II, p. 99.

‡ See E. K. Haviland, loc. cit., II, p. 100.

§ See the references on distribution functions cited in the Introduction.

ON THE COMPLETENESS OF LAMBERT FUNCTIONS*

BY EINAR HILLE AND OTTO SZÁSZ

1. *Introduction.* In the theory of Lambert series one encounters the following two sets of functions

$$S_1\colon \frac{(1-t)t^n}{1-t^n}, \quad \text{and} \quad S_2\colon \frac{(1-t)^2t^n}{(1-t^n)^2}, \quad (n = 1, 2, 3, \cdots).$$

Both sets evidently belong to the spaces $L_p(0, 1)$, $p \geqq 1$, and to $C[0, 1]$, the space of functions continuous in the closed interval $[0, 1]$, but nothing definite seems to be known about their closure properties in these various spaces. Wiener's work on the Lambert-Tauber theorem† suggests that they are complete in $L_1(0, 1)$. On the other hand, a direct elementary proof of this fact would form a first step towards a simple proof of his theorem.

In the present note we shall prove completeness in $L_p(0, 1)$, $p \geqq 1$, of sets of functions of the type

$$S(\alpha, \beta, \lambda_n)\colon \qquad (1-t)^\alpha \sum_{m=1}^{\infty} m^\beta t^{m\lambda_n}, \qquad (n = 1, 2, 3, \cdots),$$

under suitable restrictions on the parameters. It will turn out in particular that the sets S_1 and S_2 are complete in any Lebesgue space. The adjunction of $f_0(t) = 1$ to $S(\alpha, \beta, \lambda_n)$ leads to sets complete in $C[0, 1]$, but the sets S_1 and S_2 turn out to be border-line cases in the discussion, and we are not able at present to prove that they can be made complete by this device.

The analysis is capable of very considerable extension. Thus we could replace the factor $(1-t)^\alpha$ by more general multipliers. A more interesting situation is encountered if we replace the coefficients m^β by quantities c_m such that $c_1 = 1$, $|c_m| \leqq Mm^\beta$, $m > 1$. Owing to the number theoretic features of the problem, which are introduced by the inversion formula of Möbius, this

* Presented to the Society, February 29, 1936.

† See Annals of Mathematics, (2), vol. 33 (1932), pp. 1–100, especially pp. 39–43, and his book, *The Fourier Integral*, 1933, pp. 112–124.

slight change in the assumptions affects the analysis strikingly. This more general problem will be treated in a later paper.

2. *Completeness in* $L_1(0, 1)$. We start with the case $p=1$, which leads to the following theorem.

THEOREM 1. *Let*

$$k_\beta(w) = \sum_{m=1}^{\infty} m^\beta w^m, \qquad (|w| < 1).$$

Let $\alpha > \max(\beta, -1)$. *Let* $\{\lambda_n\}$ *be a set of complex numbers such that* (i) $\Re(\lambda_n) \geqq \delta > 0$, *and* (ii) $\sum_{n=1}^{\infty} \Re(1/\lambda_n) = +\infty$. *Then the set*

$$S(\alpha, \beta, \lambda_n): \qquad (1-t)^\alpha k_\beta(t^{\lambda_n}), \qquad (n = 1, 2, 3, \cdots),$$

is complete in $L_1(0, 1)$.

PROOF. A set of functions $f_n(t) \epsilon L_1(0, 1)$ is complete in this space if and only if the only bounded measurable function $g(t)$ satisfying

$$\int_0^1 f_n(t) g(t) dt = 0, \qquad (n = 1, 2, 3, \cdots),$$

is $g(t) \sim 0$. Let us apply this criterion to the set $S(\alpha, \beta, \lambda_n)$. Let $z = x + iy$, and form the function

$$F(z) = \int_0^1 k_\beta(t^z)(1-t)^\alpha g(t) dt \tag{1}$$

for $x > 0$. If $|g(t)| \leqq g$,

$$|F(z)| \leqq g \int_0^1 k_\beta(t^z)(1-t)^\alpha dt.$$

If $\beta < -1$, $k_\beta(u)$ is bounded for $0 \leqq u \leqq 1$. Since $\alpha > -1$, $|F(z)|$ is clearly bounded for $x \geqq 0$. If $\beta > -1$, we can find a constant M_β such that $k_\beta(u) \leqq M_\beta(1-u)^{-\beta-1}$. Hence

$$|F(z)| \leqq g M_\beta \int_0^1 (1-t^z)^{-\beta-1}(1-t)^\alpha dt$$

$$= g M_\beta \Gamma(\alpha+1) \sum_{m=0}^{\infty} (-1)^m \binom{-\beta-1}{m} \frac{\Gamma(mx+1)}{\Gamma(mx+\alpha+2)}.$$

The series converges uniformly for $x \geqq \delta > 0$ since $\alpha > \beta$. Replacing the binomial coefficient by $1/m$ we obtain a majorant valid for the case $\beta = -1$. Thus in all cases $F(z)$ exists as a holomorphic function of z for $x > 0$, bounded for $x \geqq \delta > 0$. In particular, it follows that the set $S(\alpha, \beta, \lambda_n)$ belongs to $L_1(0, 1)$ since we can take $g(t) = 1$, $z = \lambda_n$.

Now suppose the existence of a $g(t)$ such that for the corresponding function $F(z)$ we have $F(\lambda_n) = 0$, $(n = 1, 2, 3, \cdots)$, where the λ_n satisfy conditions (i) and (ii). Since $F(z)$ is bounded and holomorphic in a half-plane, it must vanish identically by the well known analog of Blaschke's theorem. Moreover, we know that condition (ii) is necessary as well as sufficient for this conclusion as long as we are dealing with a bounded function whose zeros satisfy (i).

But $F(z)$ is the transform of the function $(1-t)^\alpha g(t)$ by the kernel $k_\beta(t^z)$. Thus the problem is reduced to the question of whether the vanishing of the transform implies the vanishing of the transformed almost everywhere. Or is it possible that such a transformation can admit of a non-trivial representation of zero? This question seems difficult to answer one way or the other if no restrictions are put upon the transformed function except the obvious requirement that the transform shall exist as an analytic function in some right half-plane. But we are limited to a particular class of functions, namely, those of the form $(1-t)^\alpha g(t)$, where $\alpha > \max(\beta, -1)$ and $g(t)$ is bounded, and for this class it turns out that the representations of zero are trivial.

Put

$$t = e^{-u}, \qquad k_\beta(t^z) = K_\beta(uz), \qquad -(1-t)^\alpha g(t)\frac{dt}{du} = G(u).$$

Then

$$F(z) = \int_0^\infty K_\beta(uz)G(u)du, \tag{2}$$

or

$$F(z) = \sum_{m=1}^{\infty} m^\beta L(mz). \tag{3}$$

Here $L(z)$ is the Laplace transform of $G(u)$, that is

$$L(z) = \int_0^\infty e^{-uz} G(u)du, \tag{4}$$

or in terms of the old variables

$$L(z) = \int_0^1 t^z(1-t)^\alpha g(t)dt. \tag{5}$$

The last formula shows that

$$|L(z)| \leqq g\Gamma(\alpha+1)\frac{\Gamma(x+1)}{\Gamma(x+\alpha+2)}, \quad (x>0). \tag{6}$$

We can interpret (3) as a functional equation for the determination of $L(z)$ in terms of $F(z)$, where the solution is to satisfy (6). The solution is given by the special inversion formula of Möbius*

$$L(z) = \sum_{n=1}^{\infty} \mu(n) n^\beta F(nz), \tag{7}$$

where $\mu(n)$ is the factor of Möbius, that is,

$$\mu(1) = 1, \qquad \mu(n) = 0$$

if n has a quadratic factor, and

$$\mu(n) = (-1)^k$$

if n is the product of k distinct primes. In order to prove (7) rigorously, suppose that (3) is satisfied by a function $L(z)$, holomorphic in $x>0$, such that (6) holds for some choice of the constant g and $\alpha>\max(\beta, -1)$. Then the double series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^\beta L(mnz) \tag{8}$$

is clearly absolutely convergent for $x>0$. Hence, if we multiply the equation

$$\sum_{m=1}^{\infty} m^\beta L(mnz) = F(nz), \quad (n = 1, 2, 3, \cdots),$$

* A. F. Möbius, Journal für die reine und angewandte Mathematik, vol. 9 (1832), pp. 105–123, and Gesammelte Werke, vol. IV, pp. 591–612.

by $\mu(n)n^\beta$ and add the results, the series so obtained is absolutely convergent, and can be rearranged at pleasure. Thus

$$\sum_{n=1}^{\infty} \mu(n)\, n^\beta F(nz) = \sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} (nm)^\beta L(nmz)$$

$$= \sum_{k=1}^{\infty} k^\beta L(kz) \sum_{d|k} \mu(d),$$

where the last sum extends over the divisors of k. It is well known that this sum is zero unless $k=1$, when it equals one. Thus the double series reduces to a single term $L(z)$. This argument* proves that (7) gives the only solution $L(z)$ of (3) satisfying (6). In particular, if $F(z)\equiv 0$ and $L(z)$ satisfies (6), then $L(z)\equiv 0$. But this implies $G(u)=0$ almost everywhere, so that $g(t)\sim 0$. This completes the proof of Theorem 1.

It should be remarked that if $F(z)$ is defined by (2), that is, $F(z)$ is the K_β-transform of a function $G(u)$ of the special form considered here, then $F(z)$ is also the unilateral Laplace transform of a function $H(u)$. We have

$$F(z) = \int_0^\infty e^{-uz} H(u)\, du, \tag{9}$$

$$H(u) = \sum_{m=1}^{\infty} m^{\beta-1} G\left(\frac{u}{m}\right). \tag{10}$$

The last equation can also be regarded as a functional equation for the determination of $G(u)$ in terms of $H(u)$. Möbius' inversion formula applies, and gives

$$G(u) = \sum_{n=1}^{\infty} \mu(n) n^{\beta-1} H\left(\frac{u}{n}\right) \tag{11}$$

as the only solution of (10) which is $O(u^\alpha)$ for small values of u. In particular, if $H(u)\sim 0$, and $G(u)$ is known to be $O(u^\alpha)$, then $G(u)\sim 0$. Thus we have an automorphism between the theories of the two functional equations (3) and (10), as is to be expected a priori.

* See a similar argument by H. von Koch, Öfversigt af Kongliga Svenska Vetenskaps-Akademiens Förhandlingar, Stockholm, 1900, No. 5, pp. 659–668. Our functional equation could have been reduced to the system of linear equations in infinitely many unknowns treated by von Koch.

3. *Other Spaces.* Similar results hold in $L_p(0, 1)$, $p>1$.

THEOREM 2. *The set* $S(\alpha, \beta, \lambda_n)$ *is complete in* $L_p(0, 1)$, $(1<p<\infty)$, *if* $\alpha>\max(\beta+1-1/p, -1/p)$, *and* $\Re(\lambda_n)\geqq\delta>0$, $\sum_{n=1}^{\infty}\Re(1/\lambda_n)=+\infty$.

The proof follows the same lines as in the case $p=1$, the main difference being that $g(t)$ is now an arbitrary function in $L_{p'}(0, 1)$, $1/p+1/p'=1$. The necessary estimates are obtained from Hölder's inequality. In particular, (6) has to be replaced by

$$|L(z)| \leqq C\left[\frac{\Gamma(px+1)}{\Gamma(px+p\alpha+2)}\right]^{1/p}, \qquad (x>0), \tag{12}$$

but the inversion formula (7) is still valid, and gives the only solution of (3) satisfying (12).

In order that $S(\alpha, \beta, \lambda_n)$ be in $C[0, 1]$ it is necessary and sufficient that $\alpha\geqq\max(\beta+1, 0)$, $\Re(\lambda_n)>0$. But this set of functions can never be complete in $C[0, 1]$ because they all vanish at $t=0$, and at $t=1$ if $\alpha>0$. This deficiency can be overcome by joining $f_0(t)=1$ to the set. In this way we get the following theorem.

THEOREM 3. *The set* $1+S(\alpha, \beta, \lambda_n)$ *is complete in* $C[0, 1]$ *if* $\alpha>\max(\beta+1, 0)$, $\Re(\lambda_n)\geqq\delta>0$, *and* $\sum_{n=1}^{\infty}\Re(1/\lambda_n)=+\infty$.

By the theorem of F. Riesz we have to show that the only function $g(t)$ of bounded variation in $[0, 1]$ such that

$$\int_0^1 f_n(t)dg(t) = 0, \qquad (n = 0, 1, 2, \cdots), \tag{13}$$

is a constant. We form

$$F(z) = \int_0^1 k_\beta(t^z)(1-t)^\alpha dg(t), \qquad g(0) = g(1),$$

where the end point condition expresses that (13) is satisfied for $n=0$. The estimates offer no particular difficulty, and lead to

$$|L(z)| \leqq C\frac{\Gamma(x+1)}{\Gamma(x+\alpha+1)}, \qquad (x>0). \tag{14}$$

Since $\alpha>\max(\beta+1, 0)$, the inversion formula applies and gives the only solution $L(z)$ satisfying (14). In particular, if $F(z)\equiv 0$ and $L(z)$ satisfies (14), then $L(z)\equiv 0$, whence

$$A(u) \equiv \int_0^u (1-t)^\alpha dg(t) = C, \tag{15}$$

a constant, at all points of continuity of the integral. Integration by parts gives

$$(1-u)^\alpha g(u) - g(0) + \alpha \int_0^u (1-t)^{\alpha-1} g(t)dt = C.$$

The last term on the left is absolutely continuous, so $g(u)$ must be absolutely continuous in every interval $0 \leqq u \leqq 1-\delta$. But then by (15) we have $g'(u) \sim 0$ and $g(u) = g(0)$ for $u < 1$. Since $g(1) = g(0)$ by assumption, we have $g(u) = g(0)$, $0 \leqq u \leqq 1$. This completes the proof of Theorem 3.

The only remaining case of interest in $C[0, 1]$ is that in which $\alpha = \max(\beta+1, 0)$. If $\beta < -1$, $\alpha = 0$, we can still prove completeness of the set $1 + S(\alpha, \beta, \lambda_n)$ if the λ_n satisfy the conditions of Theorem 3. The case $\beta = -1$, $\alpha = 0$, is excluded, but for $\beta > -1$, $\alpha = \beta + 1$ our method fails. Take the simplest and most interesting case $\alpha = 1$, $\beta = 0$, that is, the set S_1 with unity adjoined. In order to prove the absolute convergence of the double series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} L(mnz)$$

we obviously need something better than (14). Now it is not difficult to prove that

$$\sum_{n=1}^{\infty} |L(nz)| \leqq \int_0^1 (1-t) \frac{t^x}{1-t^x} |dg(t)|, \quad (x > 0).$$

For fixed x the multiplier of the integrator is a monotone increasing function of t, whence we conclude that the integral exists and is $O(1/x)$. But this is clearly not enough for the convergence of the double series, so we do not know whether or not the inversion formula applies. It is perhaps of some interest to remark in this connection that the solution of the functional equation

$$\sum_{n=1}^{\infty} L(nz) = \frac{1}{z} \quad \text{is} \quad L(z) = \frac{1}{z} \sum_{n=1}^{\infty} \frac{\mu(n)}{n}$$

according to the inversion formula. By Landau's theorem the

second series is identically zero, so the inversion formula does not give an actual solution. Under these circumstances we are forced to leave the question of the completeness of S_1+1 in $C[0, 1]$ unanswered.*

YALE UNIVERSITY AND MASSACHUSETTS INSTITUTE OF TECHNOLOGY

GROUPS OF MOTIONS IN CONFORMALLY FLAT SPACES

BY JACK LEVINE

1. *Introduction.* In this paper we consider the problem of determining the conditions which a conformally flat space must satisfy in order that it may admit a group of motions. These conditions are expressed in Theorem 1. Conformally flat spaces admitting simply transitive groups of motions are considered in the last section. All summations are from 1 through n unless otherwise indicated.

2. *Killing's Equations.* The equations for determining the possible existence of groups of motions in a metric space are known as Killing's equations and are given by†

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} = 0. \tag{1}$$

If V_n is conformally flat, there exists a coordinate system in which $g_{ij} = e_i \delta_j^i h^2$, where $e_i = \pm 1$. In this coordinate system (1) reduce to

$$e_i \frac{\partial \xi^i}{\partial x^j} + e_j \frac{\partial \xi^j}{\partial x^i} = 0, \qquad (i \neq j,\ i, j \text{ not summed}), \tag{2}$$

$$\xi^k \frac{\partial H}{\partial x^k} + \frac{\partial \xi^i}{\partial x^i} = 0, \qquad (i \text{ not summed},\ H = \log h). \tag{3}$$

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* The completeness of $1+S(\beta+1, \beta, \lambda)$ in $C[0, 1]$ is proved for $-1 < \beta \leqq 2$ in a paper to appear in the Annals of Mathematics.

† L. P. Eisenhart, *Riemannian Geometry*, p. 234.

From (2) we obtain

$$e_i \frac{\partial^2 \xi^i}{\partial x^i \partial x^k} = - e_j \frac{\partial^2 \xi^j}{\partial x^i \partial x^k} = e_k \frac{\partial^2 \xi^k}{\partial x^i \partial x^j} = - e_i \frac{\partial^2 \xi^i}{\partial x^j \partial x^k},$$

$(i, j, k \neq$, no summing),

which gives

$$\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0, \qquad (i, j, k \neq). \tag{4}$$

From (3) we have

$$\frac{\partial \xi^i}{\partial x^i} = \frac{\partial \xi^j}{\partial x^j} \equiv \tau, \qquad (i, j \text{ not summed}). \tag{5}$$

From (2) follow

$$e_i \frac{\partial^2 \tau}{\partial x^i \partial x^j} + e_j \frac{\partial^2 \tau}{\partial x^j \partial x^i} = 0,$$

$$e_j \frac{\partial^2 \tau}{\partial x^k \partial x^k} + e_k \frac{\partial^2 \tau}{\partial x^j \partial x^j} = 0,$$

$$e_k \frac{\partial^2 \tau}{\partial x^i \partial x^i} + e_i \frac{\partial^2 \tau}{\partial x^k \partial x^k} = 0, \qquad (i, j, k \text{ not summed}),$$

so that $\partial^2\tau/\partial x^i \partial x^i = 0$, ($i$ not summed), and by (4) we get $\partial^2\tau/\partial x^i \partial x^j = 0$, so finally we have

$$\tau = \frac{\partial \xi^i}{\partial x^i} = a_0 + a_j x^j, \qquad (i \text{ not summed}), \tag{6}$$

where the a's are constants. The general solution ξ^i of (2) and (5) is found to be

$$\xi^i = b^i + a_0 x^i + x^i a_j x^j - \frac{1}{2} a_i e_i e_j (x^j)^2 + b_j^i x^j. \tag{7}$$

The a's and b's are arbitrary, with $e_i b_j^i + e_j b_i^j = 0$, ($i$, j not summed). The group generated by the ξ^i of (7) is the general conformal group of $(n+1)(n+2)/2$ parameters.* In order to define a group of motions, the ξ^i must satisfy the further conditions (3).

* S. Lie, *Theorie der Transformationsgruppen*, vol. 3, pp. 334, 347.

If we substitute the value of ξ^i as given by (7) into (3), we tain an equation which can be written as

$$\sum_{i=1}^{N} A_i u^i = 0, \qquad (N = (n+1)(n+2)/2), \tag{8}$$

where the A_i represent the N constants in the expression for ξ^i and u^i are functions of the x's. From (8) we obtain an infinite sequence of equations

$$\sum_{i=1}^{N} A_i \frac{\partial^t u^i}{\partial x^{a_1} \cdots \partial x^{a_t}} = 0, \qquad (t = 0, 1, 2, \cdots), \tag{9}$$

which must be identically satisfied in the x's. The function H being assumed analytic in a certain domain of the variables x, we may express the u^i in the form

$$u^i = u_0^i + u_j^i x^j + \frac{1}{2!} u_{jk}^i x^j x^k + \cdots,$$

and substituting for u^i in (9), we see that

$$\sum_{i=1}^{N} A_i u^i_{a_1 \cdots a_t} = 0, \qquad (t = 0, 1, \cdots). \tag{10}$$

Hence a necessary and sufficient condition for the existence of non-zero solutions for A_i is that the rank of the matrix

$$\| u_0^i, u_j^i, u_{jk}^i, \cdots \|$$

be $\leqq N-1$. Since this condition must hold for every point in the domain of analyticity, we can replace the above matrix by

$$\left\| u^i, \frac{\partial u^i}{\partial x^j}, \frac{\partial^2 u^i}{\partial x^j \partial x^k}, \cdots \right\|.$$

This matrix in turn can be replaced by the finite matrix

$$\left\| u^i, \frac{\partial u^i}{\partial x^j}, \cdots, \frac{\partial^{N-1} u^i}{\partial x^{a_1} \cdots \partial x^{a_{N-1}}} \right\|, \tag{11}$$

since we cannot have more than N independent equations (10).*

* Equations of the type (8) have been considered by M. S. Knebelman, who has obtained necessary and sufficient conditions for the existence of constant solutions A.

The u^i involve derivatives of H and the condition on the rank of the matrix (11) gives us the required restrictions on H. We state this in the following theorem.

THEOREM 1. *Given a function H defining a conformally flat space, a necessary and sufficient condition that this space admit a group of motions is that the rank of the matrix* (11) *be* $\leqq N-1$, *where* $N=(n+1)(n+2)/2$. *If the rank is* $N-r$, *the space admits a* G_r *of motions.*

We shall find the conditions for a conformally flat space with $h^2=1/f(r^2)$ to admit a group of motions, where

$$r^2=\sum e_i(x^i)^2.$$

For this case (3) becomes

$$\frac{f'}{f}=\frac{a_0+a_ix^i}{\sum e_ib^ix^i+(2a_0+a_ix^i)r^2/2}.$$

In order for the right member of the above equation to be a function of r^2, we must have

$$a_i=b^i=0, \quad \text{or} \quad a_0=0, \quad e_ib^i=ca_i, \qquad (i \text{ not summed}).$$

For the second case we find that $f(r^2)$ must be of the form

$$f(r^2)=(\alpha r^2+\beta)^2, \tag{12}$$

that is, the space is of constant curvature.* For the first case f must be of the form $f(r^2)=\alpha r^2$, where α is a constant. This proves the following theorem.

THEOREM 2. *The only metric spaces with quadratic form* $[1/f(r^2)]\sum e_i(dx^i)^2$ *which admit a group of motions are spaces of constant curvature, where f has the form* (12), *and spaces with the quadratic form* $(1/\alpha r^2)\sum e_i(dx^i)^2$. *The r^2 has the value* $r^2=\sum e_i(x^i)^2$.

It can be shown that if $H=\alpha+\alpha_ix^i$, the corresponding space admits a group of motions of at least $n-1$ parameters, and if $H=\alpha+(1/2)\alpha_{ij}x^ix^j$, the corresponding space admits a group of motions of at least n parameters, the α's being arbitrary. This is shown by a consideration of the rank of (11) for these choices of H.

* *Riemannian Geometry*, p. 85.

3. *Simply Transitive Groups.* If a motion in V_n is to be a translation, we must have $e^{2H}\sum e_i(\xi^i)^2=c$, where c is a constant. If we calculate $\partial H/\partial x^i$ from this relation and substitute in (3), it is found that $\tau=0$, and hence a necessary condition for a translation is that ξ^i be of the form $\xi^i=b^i+b^i_j x^j$. Suppose V_n admits a simply transitive group G_n of translations. Then from (3) and the fact that $\tau=0$, it is easily seen that H is constant, that is, the V_n is a flat space. This gives us the following theorem.

THEOREM 3. *If a conformally flat space admits a simply transitive group of translations, the space is flat.*

If the e_i are all the same sign, it is easily shown that the group must be Abelian.

A simply transitive group G_n being given, we consider the conditions under which there exists a conformally flat space admitting G_n as a group of motions. Necessary and sufficient conditions are given by

$$\frac{\partial H}{\partial x^i} = L_i, \tag{13}$$

$$e_i L^i_{jk} + e_j L^j_{ik} = 0, \qquad (i, j \text{ not summed, } i \neq j), \tag{14}$$

where

$$L^i_{jk} = -\xi_{a|k}\frac{\partial \xi^i_{a|}}{\partial x^j}, \qquad L_i = L^1_{1i} = \cdots = L^n_{ni}, \tag{15}$$

$$(\xi_{a|k}\xi^k_{b|} = \delta^a_b,\ \xi_{a|i}\xi^j_{a|} = \delta^j_i).$$

From (14) and (15) it follows that (13) are completely integrable. Now if ξ^i satisfies (14) and (15) we have seen it must be of the form (7), and hence we have the following theorem.

THEOREM 4. *A given simply transitive group will be a group of motions of a conformally flat space if and only if the group is a sub-group of the general conformal group. The determination of the space depends on one arbitrary constant.*

NORTH CAROLINA STATE COLLEGE

ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS*

BY H. L. KRALL

1. *Introduction.* A set $\phi_0(x) = 1, \phi_1(x), \phi_2(x)$ of polynomials of degrees 0, 1, 2, $\cdots$, is called a set of orthogonal polynomials if they satisfy

$$\int_a^b \phi_m(x)\phi_n(x)d\psi(x) = 0, \qquad \int_a^b d\psi(x) > 0, \qquad (m \neq n),$$

where $\psi(x)$ is a non-decreasing function of bounded variation. There is no restriction in assuming the highest coefficient 1.

It has been shown by W. Hahn† that if the derivatives also form a set of orthogonal polynomials, then the original set were Jacobi, Hermite, or Laguerre polynomials. His method consisted in showing that the polynomials satisfy a differential equation of the type

$$(a + bx + cx^2)\phi_n'' + (d + ex)\phi_n' + \lambda_n\phi_n = 0.$$

From this it followed that the set were Jacobi, Hermite, or Laguerre polynomials.

Here we propose to give a new proof of this result, our point of view being to answer the question: What conditions on the weight function result from assuming that both $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are sets of orthogonal polynomials? However, we shall assume that (a, b) is a finite interval and $d\psi(x) = p(x)dx$,‡ where the weight function is L-integrable.

2. *A Relation for the Weight Function* $q(x)$. Let the set $\{\phi_n'(x)\}$ be orthogonal in the interval (c, d), infinite or not, with the weight function $q(x)$, that is,

$$\int_c^d q(x)\phi_n'(x)\phi_m'(x)dx = 0, \qquad (m \neq n).$$

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* Presented to the Society, April 11, 1936.

† W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

‡ Professor Shohat informs me that he has discussed the general case $d\psi(x)$.

The polynomials $\{\phi_n(x)\}$, $\{\phi_n'(x)\}$ satisfy the recurrence relations,*

$$\phi_{n+2}(x) = (x - c_{n+2})\phi_{n+1}(x) - \lambda_{n+2}\phi_n(x),$$

$$\frac{1}{n+2}\phi_{n+2}'(x) = \frac{x - c_{n+2}'}{n+1}\phi_{n+1}'(x) - \lambda_{n+2}'\phi_n'(x),$$

$$(n \geqq 0;\ c_n,\ c_n',\ \lambda_n,\ \lambda_n' \text{ constants}).$$

Differentiating both sides of the first relation and eliminating the term containing x, by means of the second relation, we get

$$\phi_{n+1}(x) = \frac{1}{n+2}\phi_{n+2}'(x) + c_{n+2}''\phi_{n+1}'(x) + \lambda_{n+2}''\phi_n'(x),$$

$$(c_n'',\ \lambda_n'' \text{ constants}).$$

Remembering that $\phi_n'(x)$, with the weight function $q(x)$, is orthogonal to any polynomial of degree $\leqq n-2$, we get

$$\int_c^d q(x)\phi_{n+1}(x)G_{n-2}(x)dx = 0, \tag{1}$$

where $G_n(x)$ is an arbitrary polynomial of degree $\leqq n$.

LEMMA. *Let $Q(x)$ be non-negative in (c, d), and such that the numbers*

$$\beta_k = \int_c^d Q(x)x^k dx, \qquad (k = 0, 1, \cdots),$$

exist, and for a certain positive integer r

$$\int_c^d Q(x)\phi_n(x)G_{n-r-1}(x)dx = 0, \quad (n = r+1, r+2, \cdots). \tag{2}$$

Then almost everywhere

$$Q(x) = \begin{cases} P_r(x)p(x) & \text{in } (a, b), \\ 0 & \text{elsewhere}, \end{cases}$$

where $P_r(x)$ is a polynomial of degree $\leqq r$.

Consider the function

* J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, p. 24.

$$R(x) = (u_0 + u_1x + u_2x^2 + \cdots + u_rx^r)p(x).$$

We determine the $\{u_i\}$ so that

$$\int_a^b R(x)x^i dx = \int_c^d Q(x)x^i dx, \qquad (i = 0, 1, \cdots, r);$$

that is, the u_i satisfy the equations

$$\begin{aligned}
\alpha_0u_0 + \alpha_1u_1 \quad &+ \cdots + \alpha_ru_r \quad = \beta_0,\\
\alpha_1u_0 + \alpha_2u_1 \quad &+ \cdots + \alpha_{r+1}u_r = \beta_1,\\
&\cdots\cdots\cdots\cdots\cdots\\
\alpha_ru_0 + \alpha_{r+1}u_1 &+ \cdots + \alpha_{2r}u_r \quad = \beta_r,
\end{aligned}$$

where

$$\alpha_k = \int_a^b x^k p(x)dx.$$

This is always possible for the determinant of this system is known to be positive.* Now†

$$0 = \int_a^b R(x)\phi_{r+i}(x)dx = \int_c^d Q(x)\phi_{r+i}(x)dx, \qquad (i = 1, 2, \cdots).$$

And then

$$\int_a^b R(x)x^i dx = \int_c^d Q(x)x^i dx, \qquad (i = 0, 1, 2, \cdots).$$

Let

$$f(x) = \begin{cases} Q(x) - R(x) \text{ in } E_1, \text{ the points where } (a, b) \text{ and } (c, d) \text{ overlap,}\\ -R(x) \text{ in } E_2, \text{ the remainder of } (a, b),\\ Q(x) \text{ in } E_3, \text{ the remainder of } (c, d). \end{cases}$$

Then

$$\int_{E_1+E_2+E_3} f(x)x^i dx = 0, \quad (i = 0, 1, 2, \cdots).$$

If (c, d) is finite, $f(x)$ must be zero almost everywhere, and our

* See Shohat, loc. cit., p. 9, formula (19).

† As a special case, take in (2), $n = r+i$; that is, $\int_c^d Q(x)\phi_{r+i}(x)G_{i-1}(x)dx = 0$, then take $G_{i-1}(x) \equiv 1$.

lemma follows. Moreover, from the above definition of $f(x)$, we conclude that $R(x)$, hence $p(x)$, $\equiv 0$ almost everywhere in E_3, so that both intervals (a, b) and (c, d) may be reduced to their common part E_1; in other words, here we may take $(c, d) \equiv (a, b)$. If (c, d) is infinite, let $A = \max(|a|, |b|)$, so that $|x| \leqq A$ in E_1+E_2, which is identical with (a, b), and let E_4 be the part of E_3 for which $|x| \geqq (1+\alpha)A$, where $\alpha > 0$ is arbitrary. If $Q(x) \geqq 0$, we have

$$(1+\alpha)^i A^i \int_{E_4} Q(x)dx \leqq \int_{E_3} Q(x)x^i dx = \left| \int_{E_1+E_2} f(x)x^i dx \right|$$

$$\leqq A^i \int_{E_1+E_2} |f(x)|\, dx,$$

$$(1+\alpha)^i \leqq \frac{\int_{E_1+E_2} |f(x)|\, dx}{\int_{E_4} Q(x)dx},$$

for all even i, which is impossible unless $Q(x)$ is zero almost everywhere in E_4. Hence (c, d) reduces to $E_1+E_3-E_4$ and in (c, d), $|x| < (1+\alpha)A$; that is, in view of the arbitrariness of $\alpha(>0)$,

$$|x| \text{ in } (c, d) \leqq A = \max(|a|, |b|),$$

which requires that $(c, d) \equiv (a, b)$. The following important result has been established: (c, d) is finite and coincides with (a, b).

With this lemma, it follows from (1) that

$$q(x) = (rx^2 + sx + t)p(x), \tag{3}$$

and we may take $c = a$, $d = b$.

3. *Existence of* $q'(x)$. Consider the function

$$S(x) = k \int_a^x (x - l)p(x)dx,$$

where k and l are such that $S(b) = 0$, $\int_a^b S(x)dx = \int_a^b q(x)dx$. An integration by parts applied to $\int_a^b S(x)\phi'_{n+1}(x)dx$ gives, since $S(a) = S(b) = 0$,

$$\int_a^b S(x)\phi'_{n+1}(x)dx = \int_a^b \phi_{n+1}(x)k(x - l)p(x)dx = 0, \quad (n \geqq 1).$$

But $q(x)$ is the weight function for the orthogonal polynomials $\{\phi_n'(x)\}$, whence

$$\int_a^b S(x)\phi_{n+1}'(x)dx = \int_a^b q(x)\phi_{n+1}'(x)dx = 0, \quad (n \geqq 1).$$

This and the relation $\int_a^b S(x)dx = \int_a^b q(x)dx$ gives

$$\int_a^b S(x)x^n dx = \int_a^b q(x)x^n dx, \qquad (n \geqq 0),$$

and then $q(x) = S(x)$ almost everywhere. Since $S(x)$ has a derivative almost everywhere, $q(x)$ has a derivative almost everywhere and

$$q'(x) = k(x - l)p(x), \qquad q(a) = q(b) = 0. \tag{4}$$

4. *Discussion of $q(x)$ and $p(x)$.* Dividing (4) by (3), we get

$$\frac{q'(x)}{q(x)} = \frac{k(x - l)}{rx^2 + sx + t}, \qquad q(a) = q(b) = 0. \tag{5}$$

We proceed to show (i) rx^2+sx+t has real zeros, (ii) $r \neq 0$. (i) Assume rx^2+sx+t has imaginary zeros. Integrating the differential equation (5), we get

$$\log q(x) = \int \frac{k(x - l)}{rx^2 + sx + t}\,dx + c,$$

$$q(x) = K(rx^2 + sx + t)^{\alpha}e^{\beta \arctan(\gamma x+\delta)},$$

$$(\alpha, \beta, \gamma, \delta, K \text{ constants}).$$

This is incompatible with $q(a) = q(b) = 0$. (ii) Assume first $r = s = 0$. Equation (5) becomes

$$q'(x) = (2\alpha x + \beta)q(x),$$

$$q(x) = Ke^{\alpha x^2+\beta x}, \qquad (\alpha, \beta, K \text{ constants}),$$

which is not zero at a and b.

Next, suppose $r = 0$, $s \neq 0$. Equation (5) gives

$$\frac{q'(x)}{q(x)} = \frac{k(x - l)}{sx + t} = \alpha + \frac{\beta s}{sx + t},$$

$$q(x) = K(sx + t)^{\beta}e^{\alpha x}, \qquad (\alpha, \beta, K \text{ constants}).$$

This cannot vanish at both end points, $x=a$ and $x=b$.

Having thus proved (i) and (ii), we set

$$rx^2+sx+t=r(x-g)(x-h), \qquad (r\neq 0,\ g,\ h \text{ real}),$$

and rewrite (5) as follows:

$$\frac{q'(x)}{q(x)}=\frac{k(x-l)}{rx^2+sx+t}=\frac{\alpha}{x-g}+\frac{\beta}{x-h},$$

whence

$$q(x)=K(x-g)^{\alpha}(x-h)^{\beta}, \qquad (K,\ \alpha,\ \beta, \text{ constants}).$$

The conditions $q(a)=q(b)=0$ demand that $g=a$, $h=b$, so that finally (disregarding inconsequential constant factors)

$$q(x)=-r(x-a)^{\alpha}(b-x)^{\beta}.$$

And then from (3)

$$p(x)=\frac{q(x)}{rx^2+sx+t}=\frac{r(x-a)^{\alpha}(b-x)^{\beta}}{r(x-a)(b-x)},$$

$$p(x)=(x-a)^{\alpha-1}(b-x)^{\beta-1},$$

and we can see that α, β are both >0.

5. *Conclusion.* Since this is the weight function for Jacobi polynomials, we have thus established the following theorem.

THEOREM. *If $\{\phi_n(x)\}$ is a set of orthogonal polynomials with the weight function $p(x)$ in the finite interval (a, b), and if we assume that the derivatives $\{\phi_n'(x)\}$ also form a set of orthogonal polynomials in a certain interval (c, d) (infinite or not), with a non-negative weight function $q(x)$, then $\{\phi_n(x)\}$ is a set of Jacobi polynomials.*

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NORMALS TO A SPACE V_n IN HYPERSPACE

BY NOLA L. ANDERSON AND LOUIS INGOLD

1. *Introduction.* The authors consider here a generalization of known results relating to the curvature vectors of a pair of mutually orthogonal curve systems on a general two-dimensional surface S_2 in hyperspace. Several phases of this generalization have been given in papers presented from time to time* but the more connected account here given seems desirable. The generalization is of a two-fold nature: first, to vector systems, or curve systems, in n dimensions, and second, to systems not necessarily orthogonal.†

The results have been presented as they are related to the usual type of n-dimensional geometry, namely, a geometry in which tangent vectors f_i to curves of parameter x^i may be obtained by differentiation from a fundamental vector or function f. The fundamental tensor g_{ij} is determined by the formulas $f_i \cdot f_j = g_{ij}$, where the product on the left is a scalar product if a vector notation is used, or an integrated product if the function notation is employed. The discriminant $|g_{ij}|$ is assumed to be different from zero.

In the present paper, certain abbreviations are used, but the notations are essentially the same as in the paper just cited. There is also a slightly wider interpretation given to the Maschke parenthesis expressions. Thus in the vector $(fa^1 \cdots a^{n-1})$ the a^k

* The paper in the form here offered is the outgrowth of work begun in two papers presented to the Society: *First normal spaces in Riemannian geometry*, by Nola L. Anderson, presented at Lawrence, Kansas, December 1, 1928, and *Invariant normals to a space S_n contained in a function space*, by Nola L. Anderson and Louis Ingold, presented at Des Moines, Iowa, December 31, 1929. All the results had been secured and the general organization had been discussed, before the sad death of Professor Ingold on January 25, 1935 (see this Bulletin, vol. 41, p. 181).

† A brief discussion of these ideas for orthogonal systems was given by L. Ingold in the paper *A symbolic treatment of the geometry of hyperspace*, Transactions of this Society, vol. 27, pp. 574–599, but the treatment was inadequate because of the necessity of restricting the discussion to a fixed point of the space. Those unfamiliar with the methods and notations may be referred to this paper, where other references will be found.

are regarded as arbitrary covariant vectors or vector fields of components a_i^k, and the parenthesis denotes $1/g^{1/2}$ times the determinant whose first column consists of the f_i and whose column headed a^k consists of the a_i^k. The more usual case in which the a^k denote functions of the coordinates and in which a_i^k denotes $\partial a^k/\partial x^i$ is of course a special case.

When n covariant vectors $a^1, a^2, \cdots, a^n$ are used, the vectors $(fa^1 \cdots a^{i-1}a^{i+1} \cdots a^n)$ will be written in the abbreviated form (fA_i). Other abbreviations will generally be clear from these two illustrations* and the context. Where other equivalent notations for the vector or symbol f are used in parenthesis expressions, the context will show how many notations of the different types are present. The expression $(f\phi)(\phi A_i)$, where the f's and ϕ's denote equivalent notations for the fundamental function or vector f, must contain just one notation ϕ and $n-1$ notations f, since the A_i stands for $n-1$ columns. The unabbreviated form would be

$$(f^1f^2 \cdots f^{n-1}\phi)(\phi a^1a^2 \cdots a^{i-1}a^{i+1} \cdots a^n).$$

The expressions

$$(f^1 \cdots f^k a^1 \cdots a^{n-k}) \quad \text{and} \quad (f^1 \cdots f^k b^1 \cdots b^{n-k})$$

may be written simply (fa) and (fb), where the context shows how many of the a's and b's are present. The angle θ between the two k-dimensional forms just given is defined by the formula†

$$\cos\theta = \frac{(fa)(fb)}{[(\phi a)^2(\psi b)^2]^{1/2}}.$$

The use of distinct notations of ϕ, ψ instead of f in the denominator is not strictly necessary here.

We need only the angles ω_{ij} between the vectors (fA_i) and (fA_j), and the angles α_{ij} between the $(n-1)$-dimensional vectors $(f^1 \cdots f^{n-1}a^i)$ and $(f^1 \cdots f^{n-1}a^j)$.

It will be convenient also to introduce notations for certain frequently occurring invariants. Thus S will denote the paren-

* The letters $f, \phi, \psi, \theta, \cdots$, with upper indices where necessary, will be used as equivalent notations for the fundamental function or vector.

† See Anderson, *The trigonometry of hyperspace*, The American Mathematical Monthly, vol. 36, pp. 517–523.

thesis $(a^1a^2 \cdots a^n)$ divided by the product of factors of the form $[(a^m\psi)^2/(n-1)!]^{1/2}$ for $m=1, 2, \cdots, n$. In some of the formulas a second set of a's: $a'^1, a'^2, \cdots, a'^n$ are used, and S' is formed from these as S above is formed from $a^1, a^2, \cdots, a^n$.

S_r will be used to denote $[(\phi A_r)^2]^{1/2}$ divided by the product of factors of the form $[(a^m\phi)^2/(n-1)!]^{1/2}$ except that the one factor $[(a^r\phi)^2/(n-1)!]^{1/2}$ for which $m=r$ is to be omitted.

The invariant corresponding to S constructed from

$$a^1 \cdots a^{r-1}a'^m a^{r+1} \cdots a^n$$

will be denoted by $S_{m'r}$.

The invariant corresponding to S_m but constructed wholly from the $a'^1, \cdots, a'^n$ will be denoted by S_m'. These invariants are easily expressed in terms of trigonometric functions in the two and three dimensional cases.*

2. *Curvature Vectors.* Let there be given any vector field $\boldsymbol{a}$, that is, a vector $\boldsymbol{a}$ which is a function of the coordinates x^i. Also consider a set of vectors (fA_i) constructed from the covariant set $a_m{}^k$ as described in the introduction. Let $\bar{\boldsymbol{a}}$ denote $\boldsymbol{a}/(\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$, so that $\bar{\boldsymbol{a}}$ is a unit vector, and let s_i denote arc length along any curve having the vector (fA_i) for a tangent vector; then $d\bar{\boldsymbol{a}}/ds_i$ will be called the curvature vector of the vector $\boldsymbol{a}$ with respect to s_i. It is evident that the curvature of $\boldsymbol{a}$ with respect to s_i is the same as the curvature of $h\boldsymbol{a}$, where h is any scalar.

The vectors $(f\phi)(\theta\phi)(\theta A_i)$ have the same direction as the vectors (fA_i), so that the curvatures of these two with respect to any direction are the same. We wish to compute the components of the curvature vectors of (fA_i) with respect to s_i which are orthogonal to all of the (fA_i), that is, normal to the space V_n. The unit vector in the direction of $(f\phi)(\theta\phi)(\theta A_i)$ is obtained by dividing by its magnitude $(n-1)![(\theta A_i)^2]^{1/2}$, and the derivative of this unit vector with respect to s_j consists of several terms all of which are tangent to V_n except

$$N_{ij} = \frac{(\theta\phi)(\theta A_i)((f\phi)A_j)}{(n-1)![(\theta A_i)^2(\theta A_j)^2]^{1/2}},$$

* They may also be interpreted trigonometrically in space of higher number of dimensions.

and these are clearly normal to V_n. They are therefore the normal components required.

We may consider in a similar way the normal components of the curvatures of the vectors $(f\phi)(\phi a^i)$ with respect to s_j. It is easily shown that the product $(f\phi)(\phi a^i)(fa^i b^1 \cdots b^{n-2}) = 0$, where the b_m^k are the components of covariant vectors for each k, but since $(f\phi)(\phi a^i)$ belongs to V_n we say that it is a *space* normal to all of the vectors $(fa^i b^1 \cdots b^{n-2})$. When a^i is a function of the coordinates and $a_m{}^i = \partial a^i/\partial x^m$, the vector $(f\phi)(\phi a^i)$ is a normal in V_n to the hypersurface $a^i = \text{constant}$. If we proceed as above, we obtain the normals*

$$\mathcal{N}_{ij} = \frac{((f\phi)A_j)(a^i\phi)}{[(n-1)!(a^i\psi)^2(\theta A_j)^2]^{1/2}}.$$

The vectors N_{ij} and $\mathcal{N}_{ij}$ will be referred to as the first and second set, respectively, of normal curvatures of the system of covariant vectors a^i. It is readily seen that $N_{ij} = N_{ji}$; however, $\mathcal{N}_{ij} \neq \mathcal{N}_{ji}$ in general, but still certain relations must exist among the normal curvatures of the second set because, as we shall show, they are linearly expressible in terms of those of the first set.

3. *Relations among the Normal Curvatures.* Introducing the factor $(a^1 a^2 \cdots a^n) \equiv (a)$ in both numerator and denominator of the expression for N_{ij}; we have

$$N_{ij} = \frac{(a)(\theta\phi)(\theta A_i)((f\phi)A_j)}{(a)(n-1)![(\theta A_i)^2(\theta A_j)^2]^{1/2}}.$$

By use of the identity

$$(a)(\theta\phi) = \sum_{r=1}^{r=n} (a^1 \cdots a^{r-1}\theta a^{r+1} \cdots a^n)(a^r\phi),$$

this reduces after some rearrangement to

$$N_{ij} = \sum_{r=1}^{r=n} \frac{(-1)^{r+1}(\theta A_r)(\theta A_i)(a^r\phi)((f\phi)A_j)}{(n-1)![(\theta A_i)^2(\theta A_j)^2]^{1/2}},$$

* Other normal curvatures, of course exist. If arc lengths of curves tangent to the vectors $(f\phi)(\phi a^i)$ are denoted by σ_i, the curvature of these vectors with respect to σ_j is clearly another example, and undoubtedly interesting relations involving these curvatures analogous to those given in the text would exist.

and this is readily reduced to

$$N_{ij} = \sum_{r=1}^{r=n} (-1)^{r+1} \frac{S_r}{S} \cos \omega_{ri} \mathcal{N}_{rj}.$$

In order to express, conversely, the $\mathcal{N}_{ij}$ in terms of the N_{ij}, we make use of the identity

$$(f\phi)(a^i\phi)(a) = \sum_{r=1}^{r=n} (-1)^{r+1} (a^i\phi)(a^r\phi)(fA_r),$$

which gives

$$\frac{(f\phi)(a^i\phi)}{[(n-1)!(a^i\phi)^2]^{1/2}} = \sum_{r=1}^{r=n} \frac{(-1)^{r+1}(a^i\phi)(a^r\phi)[(\theta A_r)^2]^{1/2}}{(a)[(n-1)!(a^i\psi)^2]^{1/2}} \cdot \frac{(fA_r)}{[(\theta A_r)^2]^{1/2}}.$$

The components normal to V_n of the derivatives with respect to s_j of the two sides of this equation are equal; hence

$$\mathcal{N}_{ij} = \sum_{r=1}^{r=n} \frac{(-1)^{r+1}(a^i\phi)(a^r\phi)[(\theta A_r)^2]^{1/2}}{(a)[(n-1)!(a^i\psi)^2]^{1/2}} N_{rj},$$

and this readily reduces to

$$\mathcal{N}_{ij} = \sum_{r=1}^{r=n} (-1)^{r+1} \frac{S_r}{S} \cos \alpha_{ri} N_{rj}.$$

In two dimensions consider the system of curves $a^i = \text{const.}$ At each point this system consists of just one pair of curves. There is a single angle $\omega_{12} = \alpha_{12} = \omega$; of course, $\omega_{11} = \omega_{22} = 0$. The two sets of formulas for N_{ij} and $\mathcal{N}_{ij}$ are

$$N_{11} = \frac{\mathcal{N}_{11} - \cos\omega \mathcal{N}_{21}}{\sin\omega}, \qquad \mathcal{N}_{11} = \frac{N_{11} - \cos\omega N_{21}}{\sin\omega},$$

$$N_{12} = N_{21} = \frac{\mathcal{N}_{12} - \cos\omega \mathcal{N}_{22}}{\sin\omega} \qquad \mathcal{N}_{12} = \frac{N_{12} - \cos\omega N_{22}}{\sin\omega},$$

$$= \frac{\cos\omega \mathcal{N}_{11} - \mathcal{N}_{21}}{\sin\omega}, \qquad \mathcal{N}_{21} = \frac{\cos\omega N_{11} - N_{21}}{\sin\omega},$$

$$N_{22} = \frac{\cos\omega \mathcal{N}_{12} - \mathcal{N}_{22}}{\sin\omega}, \qquad \mathcal{N}_{22} = \frac{\cos\omega N_{12} - N_{22}}{\sin\omega}.$$

The following are sample formulas in three dimensions:

$$N_{11} = (1/S)(\sin A\,\mathcal{N}_{11} - \sin B\cos\gamma\,\mathcal{N}_{21} + \sin C\cos\beta\,\mathcal{N}_{31}),$$

$$\mathcal{N}_{11} = (1/S)(\sin A N_{11} - \sin B\cos C N_{21} + \sin C\cos B N_{31}).$$

4. *Normal Curvatures of two Systems.* In this section we obtain formulas connecting the normal curvatures of the first set belonging to a system a^i with those belonging to another system a'^i. We multiply numerator and denominator of the expressions for N_{ij} by the square of $(a'^1 a'^2 \cdots a'^n)$,

$$N_{ij} = \frac{(\theta\phi)(\theta A_i)((f\phi)A_j)(a')^2}{(a')^2[(\theta A_i)^2(\theta A_j)^2]^{1/2}(n-1)!}.$$

By making use of the identities

$$((f\phi)A_j)(a') = \sum_{r=1}^{r=n}(-1)^{r+1}(a'^r A_j)((f\phi)A'_r),$$

$$(\theta A_i)(a') = \sum_{m=1}^{m=n}(-1)^{m+1}(a'^m A_i)(\theta A'_m),$$

we have

$$\begin{aligned} N_{ij} &= \sum_{r=1}^{r=n}\sum_{m=1}^{m=n}\frac{(-1)^{r+m}(\theta\phi)(a'^r A_j)(a'^m A_i)(\theta A'_m)((f\phi)A'_r)}{(a')^2[(\theta A_i)^2(\theta A_j)^2]^{1/2}(n-1)!} \\ &= \sum_{r=1}^{r=n}\sum_{m=1}^{m=n}\frac{(-1)^{r+m}(a'^r A_j)(a'^m A_i)[(\theta A'_m)^2(\theta A'_r)^2]^{1/2}}{(a')^2[(\theta A_i)^2(\theta A_j)^2]^{1/2}}N'_{rm} \\ &= \sum_{r=1}^{r=n}\sum_{m=1}^{m=n}(-1)^{r+m}\frac{S_{r'),j}S_{m'),i}S'_m S'_r}{S'^2 S_i S_j}N'_{rm}. \end{aligned}$$

This is the formula desired. It is a generalization of the formulas connecting the normal curvatures of two orthogonal families of curves on a two dimensional surface with those of a second orthogonal family. It is a generalization not only to n dimensions, but also to systems not necessarily orthogonal. A similar procedure leads to the formula

$$\mathcal{N}_{ij} = (-1)^{r+m}\frac{S_{r'),j}S'_{i),m'}S'_r}{S_j S'^2}\mathcal{N}'_{mr}.$$

Consider the case of two curves on a surface, $a=$const., $b=$const., meeting at an angle ω, and let $a'=$const. pass through their point of intersection P making an angle α with $a=$const.

Also let $b'=$const. pass through P making an angle β with $a=$const. Then we may write

$$N'_{12} = \frac{\sin(\omega-\alpha)\sin(\omega-\beta)}{\sin^2\omega} N_{11} + \frac{\sin\alpha\sin(\omega-\beta)+\sin\beta\sin(\omega-\alpha)}{\sin^2\omega} N_{12} + \frac{\sin\alpha\sin\beta}{\sin^2\omega} N_{22}.$$

This formula gives the normal curvatures of any two curves through P in terms of the normal curvatures of any given pair of curves $a=$const., $b=$const., and the angles which the different curves make with $a=$const. The same formula takes care of all cases. Thus to obtain N'_{11} make $\beta=\alpha$; to obtain N'_{22} make $\beta=\alpha$.

To obtain the usual formulas for two orthogonal systems make $\omega=\pi/2$, and then in turn (1) $\beta=\alpha$, (2) $\beta=\alpha+\pi/2$, (3) place $\beta=\alpha$ and then replace α by $\alpha+\pi/2$. We now have the following known formulas.*

(1) $N'_{11} = \cos^2\alpha N_{11} + 2\sin\alpha\cos\alpha N_{12} + \sin^2\alpha N_{22}$,

(2) $N'_{12} = \sin\alpha\cos\alpha(N_{22}-N_{11}) + (\cos^2\alpha-\sin^2\alpha)N_{12}$,

(3) $N'_{22} = \sin^2\alpha N_{11} - 2\sin\alpha\cos\alpha N_{12} + \cos^2\alpha N_{22}$.

We mention finally that in this general two-dimensional formula for N'_{12}, if α is kept constant while β varies, the formula

$$N'_{12} = \frac{\sin(\omega-\beta)}{\sin^2\omega}[\sin(\omega-\alpha)N_{11}+\sin\alpha N_{12}] + \frac{\sin\beta}{\sin^2\omega}[\sin(\omega-\alpha)N_{12}+\sin\alpha N_{22}]$$

shows that N_{12} lies in the plane of the two fixed vectors

$$\sin(\omega-\alpha)N_{11}+\sin\alpha N_{12} \quad \text{and} \quad \sin(\omega-\alpha)N_{12}+\sin\alpha N_{22}.$$

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* See Wilson and Moore, *Differential geometry of two dimensional surfaces in hyperspace*, Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), pp. 270–364.

A CHARACTERIZATION OF MANIFOLD BOUNDARIES IN E_n DEPENDENT ONLY ON LOWER DIMENSIONAL CONNECTIVITIES OF THE COMPLEMENT*

BY R. L. WILDER

In my recent paper *Generalized closed manifolds in n-space*† it was shown‡ that a compact point set B in E_n, common boundary of (at least) two domains D and D_1 which are respectively u.l.i-c.§ for $0 \leqq i \leqq j$ and $0 \leqq i \leqq n-j-3$ (where $n-2 > j \geqq (n-3)/2$), and such that the Betti numbers $p^{i+1}(D)$, $p^{i+2}(D)$, $\cdots$, $p^{n-2}(D)$ are finite, is a g.c.$(n-1)$-m. This constituted a generalization of a former result‖ to the effect that when $n=3$, D and D_1 are u.l.0-c., and $p^1(D)$ is finite, B is a closed 2-manifold. In the present note I propose to show, as principal result, that the above conditions on the numbers $p^{i+2}(D)$, $\cdots$, $p^{n-2}(D)$ are irrelevant, and furthermore that it is immaterial whether we place the restriction as to finiteness on $p^{i+1}(D)$ or on $p^{n-i-2}(D_1)$. It turns out that the only essential requirements are that the upper limits on the dimensions for which D and D_1 are u.l.i-c. must total at least $n-3$, and that one of the domains have a finite Betti number as just stated.

For the sake of brevity we make the following definitions. We shall understand without explicit statement hereafter that the imbedding space is $E_n (n \geqq 3)$ (euclidean space of n dimensions).

DEFINITION. A metric space will be said to be *completely i-avoidable*¶ at a point P if for every $\epsilon > 0$ there exist δ and η, $\epsilon > \delta > \eta > 0$, such that if γ^i is a cycle on $F(P, \delta)$, then $\gamma^i \sim 0$ on $S(P, \epsilon) - S(P, \eta)$.

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* Presented to the Society, December 29, 1934.

† Annals of Mathematics, vol. 35 (1934), pp. 876–903; to be referred to hereafter as G.C.M.

‡ Principal Theorem E of G.C.M.

§ u.l.i-c. = uniformly locally i-connected; see G.C.M. for definition.

‖ R. L. Wilder, *On the properties of domains and their boundaries in E_n*, Mathematische Annalen, vol. 109 (1933), pp. 273–306, Theorem 20; to be referred to hereafter as D.B.

¶ See condition (3), definition M^n, of G.C.M.

THEOREM 1. *Let M be the boundary of a u.l.i-c. domain D, $(0 \leqq i \leqq n-j-2)$, and P a point of M at which M is completely $(n-j-2)$-avoidable. Then there exists for every $\epsilon>0$ an $\eta>0$ such that if $\gamma^i \subset S(P, \eta)$ links M, then γ^i is linked with a cycle Γ^{n-j-1} of $D \cdot S(P, \epsilon)$, and with a cycle Γ_1^{n-j-1} of $M \cdot S(P, \epsilon)$.**

PROOF. Let ϵ' be an arbitrary positive number $<\epsilon$, and let δ and η be such that a γ^{n-j-2} of $M \cdot F(P, \delta)$ is homologous to zero on $M \cdot [S(P, \epsilon') - S(P, \eta)]$. Suppose $\gamma^i \subset S(P, \eta)$ links M. Let $H = M \cdot \overline{S(P, \delta)}$ and $K = M \cdot \overline{S(P, \epsilon')}$. Then γ^i links K. For suppose not. Then there exists a chain $C_1^{i+1} \rightarrow \gamma^i$ in $E_n - K$, and hence in $E_n - H$. A chain $C_2^{i+1} \rightarrow \gamma^i$ in $S(P, \eta)$ lies also in $E_n - \overline{(M-H)}$. Then the cycle $C_1^{i+1} - C_2^{i+1}$ must link $H \cdot \overline{M-H}$, else by the Alexander Addition Theorem γ^i does not link $H + \overline{M-H} = M$. But then $C_1^{i+1} - C_2^{i+1}$ is linked with an $(n-j-2)$-cycle of $M \cdot F(P, \delta)$, since $H \cdot \overline{M-H} \subset M \cdot F(P, \delta)$. But such a cycle bounds on $M \cdot [S(P, \epsilon') - S(P, \eta)] \subset E_n - |C_1^{i+1} - C_2^{i+1}|$,† and we have a contradiction. Thus γ^i links K.

As γ^i links K, it is linked with a cycle Γ_1^{n-j-1} of K. Since D is u.l.i-c. for $0 \leqq i \leqq n-j-2$, there lies in $D \cdot S(P, \epsilon)$‡ a cycle Γ^{n-j-1} approximating Γ_1^{n-j-1} and linked with γ^i.

THEOREM 2. *Let the compact point set M be the common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Also, let (n_1+1)-cycles of D_1 of diameter less than some fixed positive number θ bound in D_1. Then M is a g.c. $(n-1)$-m.*§

PROOF. CASE 1. Suppose $n_1 \geqq n_2$. By Theorem 3 of G.C.M., M is completely i-avoidable at all points, for $0 \leqq i \leqq n_2$. We first prove that D_1 is u.l.(n_1+1)-c. If D_1 is not u.l.(n_1+1)-c., there exist $P \subset M$ and $\epsilon>0$ such that for each $\eta>0$ there exists a

* Theorem 1 is a generalization of Theorem 4 of my paper *Concerning a problem of K. Borsuk*, Fundamenta Mathematica, vol. 21 (1933), pp. 156–167. It should be noted that the neighborhoods $S(P, \epsilon)$ are relative to E_n, not merely to M.

† If L is a chain, by $|L|$ we denote the set of points on L.

‡ See Lemma 2a of G.C.M. (A typographical error occurs in the statement of the lemma; the last "j" should be "1".)

§ Theorem 2 is an exact generalization of Theorem 8 of the paper in Fundamenta Mathematica, vol. 21, cited above.

cycle ${}^{n_1+1}\gamma$ in $D_1 \cdot S(P, \eta)$ that does not bound in $D_1 \cdot S(P, \epsilon)$. However, let us choose δ and η to satisfy the complete i-avoidability requirement with $\eta < \theta$. By hypothesis, there exists $K_1^{n_1+2} \to {}^{n_1+1}\gamma$, in D_1 and hence (for $i = n_2$) in $E_n - H$ (H as defined in proof of Theorem 1). Any $K_2^{n_1+2} \to {}^{n_1+1}\gamma$ in $S(P, \eta)$ also lies in $E_n - [F(P, \epsilon) + \overline{M - H}]$. Then $K_1^{n_1+2} - K_2^{n_1+2}$ must link a cycle of $M \cdot F(P, \delta)$, else ${}^{n_1+1}\gamma$ bounds in $D_1 \cdot S(P, \epsilon)$. But then it is linked with a Γ^m of $M \cdot F(P, \delta)$, where $M = n - (n_1+2) - 1 = n_2$; such a cycle, however, bounds on $M \cdot [S(P, \epsilon) - S(P, \eta)]$, hence in $E_n - (K_1^{n_1+2} - K_2^{n_1+2})$. Thus the supposition that ${}^{n_1+1}\gamma$ does not bound in $D_1 \cdot S(P, \epsilon)$ leads to a contradiction.

We may now show that D_1 is u.l.i-c. for $n_1 + 2 \leqq i \leqq n-2$. Let j be such a fixed value of i; we note that $n_2 \geqq n - j - 1 \geqq 1$. Suppose D_1 not u.l.j-c. Then we may determine a point P of M and an $\epsilon > 0$ such that for each $\eta > 0$ there is a cycle ${}^j\gamma$ of $D_1 \cdot S(P, \eta)$ that fails to bound in $D_1 \cdot S(P, \epsilon)$. Let δ and η be such that (1) $\epsilon > \delta > \eta > 0$, (2) any $(n-j-2)$-cycle of $M \cdot S(P, \delta)$ bounds in $M \cdot [S(P, \epsilon) - S(P, \eta)]$, (3) any $(n-j-1)$-cycle of $D_2 \cdot S(P, \delta)$ bounds in $D_2 \cdot S(P, \epsilon)$ and hence in D_2, and (4) if an ${}^j\gamma$ links M, then (Theorem 1) it is linked with an $(n-j-1)$-cycle of $D_2 \cdot S(P, \delta)$. Now if an ${}^j\gamma$ of D_1 were linked with M, we could by condition (4) determine an $(n-j-1)$-cycle of $D_2 \cdot S(P, \delta)$ with which ${}^j\gamma$ is linked. As this would not be possible by condition (3), we can suppose that ${}^j\gamma$ does not link M. Then there exists a chain $K_1^{j+1} \to {}^j\gamma$ in $E_n - M$, hence in $E_n - M \cdot \overline{S(P, \delta)}$. Let K_2^{j+1} be an arbitrary chain of $S(P, \eta)$ bounded by ${}^j\gamma$, and we have $K_2^{j+1} \to {}^j\gamma$ in $E_n - [F(P, \epsilon) + M - M \cdot S(P, \delta)]$. As before, we see by applying condition (2) that ${}^j\gamma$ bounds in $D_1 \cdot S(P, \epsilon)$.

Thus D_1 is u.l.i-c. for $0 \leqq i \leqq n-2$, and for this case the theorem follows from Principal Theorem C of G.C.M.*

CASE 2. Suppose $n_1 < n_2$. In this case we show that D_2 is u.l.i-c. for $n_2 + 1 \leqq i \leqq n-2$. We note that M is completely $(n-j-2)$-avoidable for $0 \leqq n-j-2 \leqq n_1$ at all points. The proof then follows the general method of Case 1.

The following corollary is obvious.

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* That D_1 is simply $(n-1)$-connected follows from the fact that M, being a common boundary of two domains, is a continuum.

COROLLARY. *A compact set that is the common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-2$, is a g.c.$(n-1)$-m.*

THEOREM 3. *Let M be a common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then if $p^{n_k+1}(D_k)$ is finite for either $k=1$ or 2, there exists a $\theta>0$ such that (n_k+1)-cycles of D_k of diameter $<\theta$ bound in D_k.**

PROOF. Take, for instance, $p^{n_1+1}(D_1)$ finite. Let $n_1+1=k$. Denote the cycles of a k-basis of D_1 by Γ_i^k,$(i=1, 2, \cdots, m)$. By the method of proof of Theorem 5 of G.C.M. we can prove the following lemma.

LEMMA. *Let D be a u.l.i-c. domain, $(0 \leqq i \leqq j)$, and let Γ_i^k, $(i=1, 2, \cdots, m; 0 \leqq n-k-1 \leqq j+1)$, be a set of independent cycles linking $\overline{D}$. Then in D there exist independent cycles γ_i^{n-k-1}, $(i=1, 2, \cdots, m)$, such that every linear combination of the Γ's is linked with at least one γ.*

Applying the lemma, we see that there exists in D_2 a set of $(n-k-1)$-cycles γ_i^{n-k-1}, $(i=1, 2, \cdots, m)$, such that every linear combination of the Γ's is linked with at least one of the γ_i^{n-k-1}. The remainder of the proof is similar to that for Theorem 14 of D.B. From Theorems 2 and 3 we have our principal result.

PRINCIPAL THEOREM. *Let a compact point set M be a common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then, if one of the numbers $p^{n_k+1}(D_k)$ is finite, M is a g.c.$(n-1)$-m.*

For the case $n=3$, where necessarily the numbers n_1 and n_2 as defined above must equal 0, I have shown in D.B. that without the single condition as to the finiteness of one of the numbers $p^{n_k+1}(D_k)$, not only may the boundary fail to be a manifold, but it may be the common boundary of three or more domains. However, if M has a single point P such that all 1-cycles of $D_k \cdot S(P, \epsilon)$ bound in D_k, then M is the common boundary

* Compare Theorem 14 of D.B.

of only two domains. (Theorem 11 of D.B.) We now extend the latter result to higher dimensions.*

THEOREM 4. *Let M be a common boundary of two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then, if for (at least) one of the values of k, there exists a point P of M and an $\epsilon>0$ such that all (n_k+1)-cycles of $D_k \cdot S(P, \epsilon)$ bound in D_k, it follows that M is the common boundary of only two domains. Indeed, at P, M is locally a g.c.$(n-1)$-m.*†

PROOF. Let $n_1 \geqq n_2$. As both D_1 and D_2 are u.l.0-c., M is a Jordan (or Peano) continuum, and the component C of $M \cdot S(P,\epsilon)$ determined by P is an open subset of M. By the method of argument used for Theorem 9 of D.B., C is the common boundary of two u.l.i-c., $(0 \leqq i \leqq n_k)$, domains D_k', $(k=1, 2)$, in $S(P, \epsilon)$, where all points of D_k' in a certain neighborhood U (rel. E_n) of C belong to D_k and conversely. As in Theorem 3 of G.C.M. we show that C is completely i-avoidable at all points for $0 \leqq i \leqq n_2$.

We may now proceed, as in Theorem 2 above, to show that one of the domains D_k' is u.l.i-c. for $0 \leqq i \leqq n-2$ at all points of C. Following this, we may show by methods such as those used to prove Theorem 12 of G.C.M. that in U there exist only points of $C+D_1+D_2$.

In conclusion we note that in higher dimensions there exist, a priori, further possibilities concerning common boundaries of several domains. For instance, does there exist for some E_n a common boundary of three domains D_k, $(k=1, 2, 3)$, such that D_k is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1>n_2>n_3$? The answer, in case $n_1+n_3 \geqq n-3$, is clearly negative by virtue of the corollary to Theorem 2 above; and indeed we must have $n_1+n_2 \leqq n-3$ in such a case. For the case $n_1+n_2=n-3$, let us consider the Betti numbers $p^{n_1+m}(E_n-M)$, where $n_1+m \leqq n-2$ and $n-(n_1+m)-1 \leqq n_3$ (if any such exist). By the proof of Theorem 4 of G.C.M. we may show $p^i(B)$ finite for $0 \leqq i \leqq n_3$. Consequently the num-

* It will be noted that we show now that the "ϵ-condition" is needed only for one domain.

† That is, conditions (2), (3) of definition M^{n-1} of G.C.M. are satisfied for some connected open neighborhood U of P, and so on.

bers $p^{n_1+m}(E_n-M)$ are all finite. Thus we have the following theorem.

THEOREM 5. *Let M be a common boundary of three distinct domains D_k, $(k=1, 2, 3)$, such that D_k is u.l.i-c. for $0 \leqq i \leqq n_k$, and $n_1 \geqq n_2 \geqq n_3$. Then $n_1+n_2 \leqq n-3$, and if there exists $m>0$ such that $n_1+m \leqq n-2$ and $n-(n_1+m)-1 \leqq n_3$, the Betti numbers $p^{n_1+m}(E_n-B)$ and $p^i(B)$, $(0 \leqq i \leqq n_3)$, are all finite.**

THE UNIVERSITY OF MICHIGAN

ON THE NORMAL RATIONAL n-IC

BY HELEN SCHLAUCH ADAMS

1. *Notation.* A point α of n-space may be represented by the binary form $(\alpha t)^n=(\alpha_1 t_1+\alpha_2 t_2)^n$ with non-symbolic coefficients $\alpha_0, \cdots, \alpha_n$. If $(\alpha t)^n$ is a perfect nth power $(t_1 t)^n$, α will be the point on C^n of S_n whose parameter is t_1, or briefly the point t_1. Also if $(at)^n$ is a binary form, all points which satisfy the linear apolarity condition $(\alpha a)^n=0$ lie on the $S_{n-1}a$ with coordinates $a_0, \cdots, a_n$. The S_{n-p} $(t_1 t)^p(\beta t)^{n-p}$, with parameters $\beta_0, \cdots, \beta_{n-p}$, is the osculating $(n-p)$-space O_{n-p,t_1} to C^n at t_1.† This notation is helpful in the development of some of the properties of the normal rational n-ic curve. Many of the analogous properties for the case $n=5$ have been found by other methods by A. L. Hjelmann.‡

2. *The Axes of C^n.* An axis of C^n is a line which lies in $(n-1)$ O_{n-1}'s to C^n. The axes of C^n are given by

$$(\alpha t)^n = (t_1 t)(t_2 t) \cdots (t_{n-1} t)(st),$$

parameters s_0, s_1, the t_i being parameters of points of C^n.

** Thus, although we have no actual example, it is conceivable that there exists, in E_4, a common boundary M of three domains D_k each of which is u.l.i-c. for $i=0$, 1. If so, $p^2(D_k)$ is infinite for $k=1$, 2, 3; and $p^3(E_4-M)$ is finite.

† Grace and Young, *The Algebra of Invariants*, 1903, Chapter 11.

‡ A. L. Hjelmann, *Sur les courbes gauches rationelles du cinquième ordre*, Annales Academiae Scientiarum Fennicae, (A), vol. 3 (1912–13), No. 11.

Any point $(zt)^n$ of the O_{n-1,t_1} determines n O_{n-1}'s whose t's are roots of $(zt)^n=0$. Those t's which are not t_1 determine the only axis through Z not in O_{n-1,t_1}: $(t_1t)(t_2t)\cdots(t_{n-1}t)(st)$. This axis determines the unique image point Z': $(t_1t)(t_2t)\cdots(t_{n-1}t)(t't)$ in $(t't)(rt)^{n-1}$, parameters $r_0, \cdots, r_{n-1}$.

Then the points of any axis to C^n of S_n and the points of C^n are in one-to-one correspondence, and there are n axes through any point Z of an O_{n-1} only one of which lies outside O_{n-1}. There is a one-to-one correspondence between the axes of C^n not in O_{n-1,t_1} and the points of $O_{n-1,t'}$ which establishes the collineation between pairs of points in two O_{n-1}'s. Similarly, there are ∞^{n-s-1} axes in an O_{n-1,t_1}, and the points of each of the ∞^{n-s} O_{n-s-1,t_1}'s not in O_{n-s,t_1} correspond one-to-one with the points of O_{n-s,t_1}.

3. *Axes in an R_j.* Any R_j meets any S_{2n-2j} in an S_{n-j} whose points correspond, by the collineation of §2, to those of an E_{n-j} in O_{2n-2j}. But S_{n-j} and E_{n-j} together fix a point P of O_{2n-2j}. Then in the two O_{2n-j}'s are determined two points P and P' which are images in the collineation, since the two points are intersections of corresponding S_{n-j}'s. But since such points lie on one axis, P and P' determine an axis which lies in R_j. *Then any R_j contains an axis of C^n, and the variety of axes from an S_j which lies in an O_{n-1} is a V_{j+1}^{j+1}.*

4. *The Osculants to C^n.* The $(i-1)$st osculant to C^n at t_1 is $(t_1t)^{n-i+1}(s_1t)(s_2t)\cdots(s_{i-1}t)$ as t_1 varies.* Then the variety of tangents to it is

$$(t_1t)^{n-i}(s_1t)(s_2t)\cdots(s_{i-1}t)(\alpha t), \text{ parameters } \alpha_0, \alpha_1.$$

This variety meets the O_{n-i} $(s_1t)(s_2t)\cdots(s_{i-1}t)(s_it)(\beta t)^{n-i}$, parameters $\beta_0, \cdots, \beta_{n-i}$, in the curve

$$C_{t_1}^{n-i}: \qquad (t_1t)^{n-i}(s_1t)(s_2t)\cdots(s_it) \text{ as } t_1 \text{ varies},$$

which is of the $(n-i)$th order. Obviously the O_s's to $C_{t_1}^{n-i+1}$ form O_{s-1}'s to $C_{t_1}^{n-i}$ and $C_{t_1}^{n-i}$ is the ith osculant of C^n at t_1.

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* The first osculant is discussed by G. Castelnuovo in *Studio dell'involuzione generale sulle curve razionali mediante la loro curva normale dello spazio a n dimensioni*, Atti Reale Istituto Veneto, (6), vol. 4 (1885–6), p. 1173; and by St. Jolles, *Die Theorie der Osculanten und des Sehnensystems der Raumcurve* IV *Ordnung* II *Species*, Aachen, 1886.

Thus the ith osculant $C_{t_1}^{n-i}$ is the locus of the points of intersection of the tangents to $C_{t_1}^{n-i+1}$ after the $n-i$ lines common to the variety of the tangents and O_{n-i} are removed. It is normal and rational.

5. *The n-ahedra in an O_{n-1,t_1} Determined by O_{n-1}'s from the Points of r.* Considerany point $Y(yt)^n$ on the line $r(\alpha y+\beta z)^n$ which is not an axis, and does not meet C^n. The n O_{n-1}'s from Y to C^n intersect the O_{n-1,s_1} in the O_{n-2} $(t_1t)(s_1t)(\alpha t)^{n-2}$, parameters $\alpha_0, \cdots, \alpha_{n-2}$, where the t_1's are roots of $(t_1y)^n=0$. Now, the n O_{n-2}'s from Y must osculate the first osculant by §4, and they form an n-ahedron in O_{n-1,s_1}. If $Z(zt)^n$ is a second point of r, the n-ahedron determined by any point of r will have vertices of the type

$$(1) \qquad \prod_{j=1}^{n-1}(\alpha t_{1,i+j}+\beta r_{1,i+j})(s_1t),$$

where the t_1's are roots of $(t_1y)^n=0$, and the r_1's are roots of $(r_1z)^n=0$.

Let $\sum a_{ik}x_ix_k'$, $(i, k=0, 1, \cdots, n)$, be a quadric V_{n-2}^2 in O_{n-1,s_1}. In order that any two n-ahedra (1) determined by α_1, β_1 and α_2, β_2 be self polar with respect to V_{n-2}^2, $n(n-1)/2+n$ conditions must be satisfied, one more than the number required to determine V_{n-2}^2. Then the two n-ahedra can be self polar with respect to one V_{n-2}^2 only if the determinant Δ of all the expressions $a_{ik}x_ix_k'$ vanishes. But if Δ is arranged so that the $n(n-1)/2$ rows which express the condition that the first n-ahedron be self polar appear first and the n rows relating to the second follow, then Δ can be so reduced that the nth and $(n-1)$st rows involve the same functions of the t_1's, r_1's, α_1, and β_1, while the last two rows involve those same functions of the t_1's, r_1's, α_2, and β_2. The differences of the elements of the two rows will then be constants in each case, and will be the same constants, so that the value of Δ is zero.

Thus the O_{n-1}'s from points of a line which is not an axis and does not meet C^n form n-ahedra in an O_{n-1,s_1}. Obviously, the vertices of all the n-ahedra determined by r form a locus K^{n-1} which is in one-to-one correspondence with $C_{t_1}^1$ and is thus of order $n-1$. Also, there exists a single quadric variety V_{n-2}^2 in O_{n-1,s_1} with re-

spect to which the n-ahedra determined by all points of r are self polar.

6. *Apolarities of Points Related to C^n.* Let $B(bt)^{n-1}$ and $D(dt)^{n-1}$ be two points of O_{n-1,s_1} which are polar with respect to V^2_{n-2}. Then if $(et)^{2n-2} \equiv (bt)^{n-1}(dt)^{n-1}$, the condition that B and D be polar is $(ae)^{2n-2} = 0$. But the $(n-1)$ O_{n-2}'s to $C^1_{t_1}$ from B and D have points of contact which are roots of $(t_1b)^{n-1} = 0$ and $(t_1d)^{n-1} = 0$, respectively. All $2n-2$ of these points are represented by $(et)^{2n-2} = 0$. Also the curve $C^1_{t_1}$ meets V^2_{n-2} in the points which are roots of $(at)^{2n-2} = 0$. And since $(ae)^{2n-2} = 0$, $(et)^{2n-2}$ and $(at)^{2n-2}$ are apolar.

Let $(t_1^{(i)}t)^n$, $(i = 1, \cdots, s)$, be s points of C^n. The O_{n-1}'s at these points meet in an S_{n-s} containing a $C^s_{t_1}$. Now, any of the ∞^{s-1} S_{n-1}'s through S_{n-s} meet C^n in points which are roots of

$$\sum_{i=0}^{s} k_i(t_1^{(i)}t)(t_1t)^{n-1} \equiv (my)^n = 0.$$

Also any point Y of S_{n-s} determines points of osculation of O_{n-1}'s whose parameters are roots of $(y_1t)^n = 0$, where s of the t_1's are the $t_1^{(i)}$'s. But since Y lies in all s $O_{n-1,t_1^{(i)}}$'s, $(my)^n = 0$.

Then there are the following apolarity relationships among points related to C^n. The binary form representing the points of osculation of the O_{n-2}'s to $C^1_{t_1}$ from any two points of O_{n-1,s_1} which are polar with respect to V^2_{n-2}, and the binary form representing the points of intersection of $C^1_{t_1}$ with V^2_{n-2} are apolar. Also, the ∞^{s-1} binary forms of the nth order representing points of intersection with C^n of S_{n-1}'s through the S_{n-s} of an sth osculant are apolar to the binary form of the sth order representing the s points of C^n which determine the sth osculant.

7. *$K^{n-1}_{t_1}$ Related to Two Bundles of S_{n-2}'s in O_{n-1,s_1}.* The ∞^{n-2} S_{n-1}'s through r meet O_{n-1,t_1} in ∞^{n-2} S_{n-2}'s forming the bundle (P_{t_1}):

$$(bt)(t_1t)(\delta t)^{n-2}, \text{ parameters } \delta_0, \cdots, \delta_{n-2}.$$

Since if r is not an axis, every O_{n-1} meets r in a point, O_{n-1,t_1} does, and the vertex of (P_{t_1}) is this point. Call it Y_{t_1}. If (P_{r_1}) is a similar bundle in O_{n-1,r_1}, an S_{n-k-1} of (P_{t_1}) will correspond to an S'_{n-k-1} of (P_{r_1}) if they are cut out by the same S_{n-k}. Then corresponding lines of (P_{t_1}) and (P_{r_1}) are

$$(t_1t)(\alpha t)^{n-2}(\delta t), \text{ parameters } \delta_0, \delta_1, \alpha\text{'s fixed},$$

and

$$(r_1t)(\alpha t)^{n-2}(\delta t), \text{ parameters } \delta_0, \delta_1, \alpha\text{'s fixed}.$$

By the projectivity of §2, Y_{t_1} is a point of $K^{n-1}_{t_1}$. If a line of O_{n-1,t_1} is defined by S_{n-2}'s which contain an axis, it meets its image in O_{n-1,r_1} in the point of $K^{n-1}_{t_1}$:

$$(s_1^{(1)}t)(s_1^{(2)}t)\cdots(s_1^{(n-2)}t)(t_1t)(r_1t).$$

Then if r is not an axis, $K^{n-1}_{t_1}$ is the locus of points of intersection of corresponding lines of (P_{t_1}) and (P_{r_1}) when the S_{n-2}'s defining the line of (P_{t_1}) contain a fundamental axis, and the vertex of (P_{t_1}) is a point of $K^{n-1}_{t_1}$. If r is an axis, it meets O_{n-1,t_1} in a point Y of $K^{n-1}_{t_1}$ which corresponds in (P_{r_1}) to Y_{r_1}, the image of Y_{t_1} of (P_{t_1}). Thus if r is an axis, all $K^{n-1}_{t_1}$'s of O_{n-1,t_1}'s not defining r are the intersections of corresponding lines of two bundles (P_{t_1}) and (P_{r_1}).

By the correspondence between S_i's of (P_{t_1}) and (P_{r_1}), it can be shown that *the bisecant lines, $\cdots$, the j-secant S_{j-1}'s, $\cdots$, the $(n-2)$-secant S_{n-3}'s of $K^{n-1}_{t_1}$ are formed by the intersections of corresponding S_2's, $\cdots$, S_j's, $\cdots$, S_{n-2}'s of (P_{t_1}) and (P_{r_1}).*

8. *The Principal $(n-2)$-ic of S_{n-i} Associated with C^n of S_n.* Let C'^n be the image curve of C^n projected upon any S_{n-i} from a vertex S_{i-1} not containing points of C^n. It is obvious that this image curve is of the nth order and that it is in one-to-one correspondence with C^n. Obviously the image in S_{n-i} of any line meeting the vertex S_{i-1} is a point.

By §2, the variety of axes to C^n is easily seen to be of order $2n-2$. Likewise, the order of the variety of axes meeting a line r of S_{i-1} which is not itself an axis, is $2n-2$. Now from any point Y of r can be passed n O_{n-1}'s to C^n which determine, $n-1$ by $n-1$, n axes through Y, and the dimensionality of the variety of axes through r is therefore 2. *Then the axes meeting r form a surface V_2^{2n-2}.* Of this surface, r is an n-fold directrix since every point of it is n-fold on the surface. Now any S_{n-1} through S_{i-1} also passes through r, which counts for n in its intersection with V_2^{2n-2}. The residual intersection of S_{n-1} with V_2^{2n-2} is then $n-2$ lines of V_2^{2n-2}; or every S_{n-1} through S_{i-1} contains $n-2$ lines of V_2^{2n-2}. It has been shown that the image of an axis through r

is a point; obviously the image in S_{n-i} of an S_{n-1} through the vertex S_{i-1} is an S_{n-i-1}. Since there are ∞^{n-i} positions of S_{n-1}'s through S_{i-1} and ∞^{n-i} S_{n-i-1}'s in S_{n-i},* then every S_{n-i-1} of S_{n-i} contains $n-2$ points of the locus of images of axes through r. Since there are ∞^1 such images, *the locus of the images in S_{n-i} of axes through any line r of the vertex of projection not an axis is a curve of the $(n-2)$nd order called a principal $(n-2)$-ic of S_{n-i} associated with C^n.* There are ∞^{2i-4} such principal $(n-2)$-ics, one for each position of r in S_{i-1}. It can be shown that *if r is an axis, the principal curve in S_{n-i} is an $(n-3)$-ic.*

9. *Projection of the $K_{t_1}^{n-1}$'s of O_{n-1,t_1}.* We see that n of the O_{n-2}'s of §4 from a point Y of r (not an axis) intersect $n-1$ by $n-1$ in vertices of an n-ahedron since each is an S_{n-1}. Thus the vertices of such an n-ahedron are images of O_{n-1,t_1} of axes to C^n which meet r. It was also shown in §4, that these vertices lie on a $K_{t_1}^{n-1}$ of O_{n-1,t_1}. Then an O_{n-1} meets the V_2^{2n-2} of axes from r in $K_{t_1}^{n-1}$, and V_2^{2n-2} has upon it ∞^1 curves, $K_{t_1}^{n-1}$. Finally, *the ∞^1 $K_{t_1}^{n-1}$'s of the O_{n-1}'s project into the principal $(n-2)$-ic of S_{n-i}.*

10. *Projection of the n-ahedra of O_{n-2} to C^{n-1}, n Odd.* It is easily seen that there is an nth order involutorial relation between the points Y and the vertices of the n-ahedra mentioned above. Then from §9 it follows that there is an nth order involution between the points of r and those of any $K_{t_1}^{n-1}$, and that there is an involution of order n defined by the points of r and those of its principal $(n-2)$-ic in S_{n-i}, (n odd). Since the O_{n-1}'s which intersect O_{n-1,t_1} in faces of an n-ahedron discussed here meet r, the resulting O_{n-2}'s determine, with r, S_{n-1}'s which meet S_{n-i} in S_{n-i-1}'s. *Then the projection in S_{n-i} of the faces of the n-ahedra are S_{n-i-1}'s which, since the original sides were O_{n-2}'s to $C_{t_1}^{n-1}$, have $(n-2)$-fold contact with $C_{t_1'}^{n-1}$, and are inscribed in the principal $(n-2)$-ic at n points of the fundamental involution† on it, n odd. (When $i=2$, the S_{n-3}'s are stationary.)*

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* For the proof that S_n contains $\infty^{(n-m)(m+1)}$ S_m's, $(m<n)$, see G. Veronese, *La superficie omaloide normale a due dimensioni e del quatro ordine dello spazio a cinque dimensioni e le sue projezioni nel piano e nello spazio ordinario*, Memorie dell'Accademia dei Lincei, (3), vol. 19 (1884), p. 347.

† For the definition and some discussion of the fundamental involution see L. Berzolari, *Sulle curve razionali di uno spazio lineare ad un numero qualunque di dimensioni*, Annali di Matematica, (2), vol. 21 (1893), pp. 1–25; A. Brill,

11. *The Projection of an Apolarity of* §6, *n Odd.* For the s points of C^n mentioned in §6 may be chosen s points of one group of the fundamental involution. In this case, the S_{n-s} containing the sth osculant passes through a point Y of r and contains s axes. By the projection into S_{n-i}, these s axes determine points of one group of the fundamental involution of the principal $(n-2)$-ic from r. Since S_{n-s} meets r, it projects by the usual method into an $S_{n-s-i+1}$ and the S_{n-1}'s through S_{n-2} project into S_{n-i-1}'s in S_{n-i}. *Then in S_{n-i}, the sth order binary form representing s points of one group of the fundamental involution on the principal $(n-2)$-ic from r is apolar to the ∞^{s-i} nth order binary forms, each representing the points of C'^n lying in S_{n-i-1}'s through the $S_{n-s-i+1}$ containing the image of any sth osculant of C^n, n odd.*

12. *Projected $(n-1)$-ahedra in S_{n-i} Associated with the Points of $K_{t_1}^{n-1}$ in an S_{n-2} of O_{n-1,t_1}.* Since $K_{t_1}^{n-1}$ is of the $(n-1)$st order, its points are given by $(s_1 t)^{n-1}(dt) \equiv (st)^n$, d fixed. Now any S_{n-2} $(ct)(\alpha t)^{n-2} \equiv (et)^n$, parameters $\alpha_0, \cdots, \alpha_{n-2}$, will meet $K_{t_1}^{n-1}$ in points whose parameters are the roots of $(se)^n = 0$, of which there are obviously $n-1$. Then also an S_{n-2} *in an O_{n-1} will meet $K_{t_1}^{n-1}$ in $n-1$ points.*

Now by the fundamental projectivity, every point of S_{n-2,t_1} is the image of an axis (§2) and since S_{n-2,t_1} contains ∞^{n-2} points, these axes form an $n-1$ dimensional variety. This variety is also composed of the ∞^1 S_{n-2}'s which are homologous to S_{n-2,t_1} in the other O_{n-1}'s; that is, which are cut out of the ∞^1 O_{n-1}'s by the axes of the variety. Among the axes are those which meet S_{n-2,t_1} in the $n-1$ points of $K_{t_1}^{n-1}$ and which must meet every other O_{n-1} in $n-1$ points (in accordance with §5, r is not an axis). These $n-1$ points where the axes from the points in S_{n-2,t_1} of $K_{t_1}^{n-1}$ meet an O_{n-1} determine that O_{n-1}. If, however, the O_{n-1,s_i} is one of those defining an axis to one of these points, then the axis lies completely in O_{n-1,s_i}. In this case, the S_{n-2} homologous to that of O_{n-1,t_1} is defined by the $n-2$ points where the other special axes meet O_{n-1,s_i} and the point of r from which O_{n-1,s_i} was drawn.

Ueber binäre Formen und die Gleichung sechsten Grades, Mathematische Annalen, vol. 20 (1882), pp. 330–357; W. Stahl, *Ueber Fundamentalinvolutionen auf rationalen Curven*, Journal für Mathematik, vol. 104 (1889), pp. 38–61.

Since each S_{n-2,s_i} passes through a point of r which lies in the vertex S_{i-1}, $S_{n-2,j}$ will project into an S_{n-i-1} of S_{n-i}. Also every S_{n-2,s_i} contains a point of the axis which connects a point of r to a point of $K_{l_i}^{n-1}$. Now this whole axis determines a point of $K_{l_1}^{n-1}$ and, since it meets r, must project into a point of the principal $(n-2)$-ic of S_{n-i} associated with r. However, the point common to this axis and S_{n-2,s_i} is on r and thus projects into no portion of S_{n-i}. Thus the images of the other $(n-2)$ S_{n-2,s_i}'s in S_{n-i} contain the same point of the principal $(n-2)$-ic of S_{n-i} associated with r. Then, finally, *the* ∞^{n-1} S_{n-2,s_i}*'s associated with the* ∞^{n-1} S_{n-2}*'s of any* O_{n-1,t_1} *project into* S_{n-i} *in* $(n-1)$*-ahedra whose faces are* S_{n-i-1}*'s, every* $n-2$ *of which meet in a point of the principal* $(n-2)$*-ic of* S_{n-i} *associated with* r.

13. *Projection of the Variety of Axes from Points of a Line of* O_{n-1,t_1}. This V_2^2 was defined in §3. Obviously, *the lines of the variety project into lines which envelop a conic.* If the line of O_{n-1,t_1} is a bisecant of $K_{l_1}^{n-1}$, the conic has two points in common with the principal $(n-2)$-ic.

14. *On the Image of an Axis.* The points determining an axis define, $n-2$ at a time, the $(n-2)$nd osculants $C_{i_1,i_2,\cdots,i_{n-2}}^2$ to each of which the axis is a tangent (§4). Now every axis not meeting the vertex of projection, S_{i-1}, determines with it an S_{i+1} which meets S_{n-1} in a straight line, the image of the axis. *Then the image of an axis, determined by* $n-1$ *points of* C^n, *is tangent to the* $n-1$ *images of the* $n-1$ *quadratic osculants determined by these points* $n-2$ *at a time.*

Since the S_{i+1} determined by the image of the axis and S_{i-1} contains an axis (§3), *any line of* S_{n-i} *may be regarded as the projection of an axis provided* $i>(n-2)/2$. The axis of which any line in S_{n-i} is the image corresponds to $n-1$ points of C^n, which project into $n-1$ points of C'^n, so that *any line of* S_{n-1} *corresponds to* $n-1$ *points of* C'^n.

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